Application Essay

The Collapse of the Tacoma Narrows Suspension Bridge: A Modern Viewpoint

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In the summer of 1940, the Tacoma Narrows suspension bridge in the state of Washington was completed and opened to traffic. Almost immediately, observers noticed that the wind blowing across the roadway would sometimes set up large vertical vibrations in the roadbed. The bridge became a tourist attraction as people came to watch and perhaps ride the undulating bridge. Finally, on November 7, 1940, during a powerful storm, the oscillations increased beyond any previously observed, and the bridge was evacuated. Soon the vertical oscillations became rotational, as observed by looking down the roadway. The entire span was eventually shaken apart by the large vibrations, and the bridge collapsed. See [1] and [2] for interesting and sometimes humorous anecdotes associated with the bridge.

The noted engineer Theodore von Karman was asked to determine the cause of the collapse. He and his coauthors [3] claimed that the wind blowing perpendicularly across the roadway separated into vortices (wind swirls) alternately above and below the roadbed, thereby setting up a periodic, vertical force acting on the bridge. It was this force that caused the oscillations. Others further hypothesized that the frequency of this forcing function exactly matched the natural frequency of the bridge, thus leading to resonance, large oscillations, and destruction, as described in equation (31), Section 3.8. For almost fifty years, resonance was blamed as the cause of the collapse of the bridge, although the von Karman group denied this, stating that "it is very improbable that resonance with alternating vortices plays an important role in the oscillations of suspension bridges" [3].

As we can see from equation (31), Section 3.8, resonance is a *linear* phenomenon. In addition, for resonance to occur, there must be an exact match between the frequency of the forcing function and the natural frequency of the bridge. Furthermore, there must be absolutely no damping in the system. It should not be surprising, then, that resonance was not the culprit in the collapse.

If resonance did not cause the collapse of the bridge, what did? Recent research provides an alternative explanation for the collapse of the Tacoma Narrows bridge. Lazer and McKenna [4] contend that nonlinear effects, not linear resonance, were the main factors leading to the large oscillations of the bridge (see [5] for a good review article). The theory involves partial differential equations. However, a simplified model leading to a nonlinear ordinary differential equation can be constructed.

The development of the model below is not exactly the same as that of Lazer and McKenna, but it results in a similar differential equation. The example is a new one, and it shows another way that amplitudes of oscillation can increase.

Consider a single vertical cable of the suspension bridge. We assume that it acts like a spring but with different characteristics in tension and compression and with no damping. When stretched, the cable acts like a spring with Hooke's constant, b, while, when compressed, it acts like a spring with a different Hooke's constant, a. We assume that the cable in compression exerts a smaller force on the roadway than when stretched the same distance, so that $0 < a < b$. Let the vertical deflection (positive direction downward) of the slice of the roadbed attached to this cable be denoted by $y(t)$, where t represents time, and $y = 0$ represents the equilibrium position. As the roadbed oscillates under the influence of an applied vertical force (due to the von Karman vortices), the cable provides an upward restoring force equal to by when $y > 0$ and a downward restoring force equal to ay when $y < 0$. This change in the Hooke's law constant at $y = 0$ provides the nonlinearity to the differential equation. We are thus led to consider the differential equation derived from Newton's second law of motion ($F = ma$),

$$my'' + f(y) = g(t),$$

where $f(y)$ is the nonlinear function given by

$$f(y) = \begin{cases} by, & \text{if } y \geq 0 \\ ay, & \text{if } y < 0, \end{cases}$$

$g(t)$ is the applied force, and m is the mass of the section of the roadway. Note that the differential equation is linear on any interval on which y does not change sign.

Now, let us see what a typical solution of this problem would look like. We will assume that $m = 1$, $b = 4$, $a = 1$, and $g(t) = \sin 4t$. Note that the frequency of the forcing function is larger than the natural frequencies of the cable in both tension and compression, so that we do not expect resonance to occur. We also assign the following initial values to y: $y(0) = 0$, $y'(0) = \alpha$, where $\alpha > 0$, so that the roadbed starts in the equilibrium position with downward velocity.

Because of the downward initial velocity and the positive applied force, $y(t)$ will initially increase and become positive. Therefore, we first solve

$$y'' + 4y = \sin 4t, \qquad y(0) = 0,\ y'(0) = \alpha.$$

The solution is the sum of the complementary solution, $y_c(t)$, and the particular solution, $y_p(t)$. It is easy to see that $y_c(t) = c_1 \cos 2t + c_2 \sin 2t$ (see Section 3.3), and $y_p(t) = -\frac{1}{12}\sin 4t$ (see Section 3.4). Thus,

$$y(t) = c_1 \cos 2t + c_2 \sin 2t - \frac{1}{12}\sin 4t.$$

The initial conditions give

$$y(0) = 0 = c_1,$$

$$y'(0) = \alpha = 2c_2 - \frac{1}{3},$$

so that $c_2 = (\alpha + \frac{1}{3})/2$. Therefore,

$$y(t) = \frac{1}{2}\left(\alpha + \frac{1}{3}\right)\sin 2t - \frac{1}{12}\sin 4t$$

$$= \sin 2t\left[\frac{1}{2}\left(\alpha + \frac{1}{3}\right) - \frac{1}{6}\cos 2t\right]$$

where we have used $\sin 4t = 2 \sin 2t \cos 2t$. We note that the first positive value of t for which $y(t)$ is again equal to zero is $t = \pi/2$. At that point, $y'(\pi/2) = -(\alpha + \frac{2}{3})$.

After $t = \pi/2$, y becomes negative, so we must now solve the new problem

$$y'' + y = \sin 4t, \qquad y\left(\frac{\pi}{2}\right) = 0,\ y'\left(\frac{\pi}{2}\right) = -\left(\alpha + \frac{2}{3}\right).$$

The solution, found as above, is

$$y(t) = \left(\alpha + \frac{2}{5}\right)\cos t - \frac{1}{15}\sin 4t$$

$$= \cos t\left[\left(\alpha + \frac{2}{5}\right) - \frac{4}{15}\sin t \cos 2t\right].$$

The next positive value of t after $t = \pi/2$ at which $y(t) = 0$ is $t = 3\pi/2$, at which point $y'(\pi/2) = \alpha + \frac{2}{15}$.

At this point, the solution has gone through one cycle in the time interval $[0, 3\pi/2]$. During this cycle, the section of the roadway started at the equilibrium with positive velocity, became positive, came back to the equilibrium position with negative velocity, became negative, and finally returned to the equilibrium position with positive velocity. This pattern continues indefinitely, with each cycle covering $3\pi/2$ time units. The solution for the next cycle is

$$y(t) = \sin 2t\left[-\frac{1}{2}\left(\alpha + \frac{7}{15}\right) - \frac{1}{6}\cos 2t\right] \text{ on } \left[\frac{3\pi}{2}, 2\pi\right],$$

$$y(t) = \sin t\left[-\left(\alpha + \frac{8}{15}\right) - \frac{4}{15}\cos t \cos 2t\right] \text{ on } [2\pi, 3\pi].$$

It is instructive to note that the velocity at the beginning of the second cycle is $(\alpha + \frac{2}{15})$, while, at the beginning of the third cycle, it is $(\alpha + \frac{4}{15})$. In fact, the velocity at the beginning of each cycle is $\frac{2}{15}$ greater than at the beginning of the previous cycle. It is not surprising, then, that the amplitude of oscillations will increase over time, since the amplitude of (one term in) the solution during any one cycle is directly proportional to the velocity at the beginning of the cycle.

It must be remembered that the model presented here is a very simplified one-dimensional model that cannot take into account all of the intricate interactions of real bridges. The reader is referred to the account by Lazer and McKenna [4] for a more complete model. More recently, McKenna [6] has refined that model to provide a different viewpoint of the torsional oscillations observed in the Tacoma bridge.

Research on the behavior of bridges under forces continues. It is likely that the models will be refined over time and new insights will be gained from the research. However, it should be clear at this point that the large oscillations causing the destruction of the Tacoma Narrows suspension bridge were not the result of resonance.