Problems of the matrix-Solution

1. Let $A=\begin{pmatrix}1 & 2 & 3\\ -1 & 0 & 2\end{pmatrix}$ and $B=\begin{pmatrix}-1 & 5 & -2\\ 2 & 2 & -1\end{pmatrix}$. Find $A+B$, $3B$, $2A+B$, $A-2B$, A^T, $(A+B)^T$.

$$A+B=\begin{pmatrix}1 & 2 & 3\\ -1 & 0 & 2\end{pmatrix}+\begin{pmatrix}-1 & 5 & -2\\ 2 & 2 & -1\end{pmatrix}=\begin{pmatrix}0 & 7 & 1\\ 1 & 2 & 1\end{pmatrix}$$

$$3B=\begin{pmatrix}-3 & 15 & -6\\ 6 & 6 & -3\end{pmatrix}$$

$$2A+B=2\begin{pmatrix}1 & 2 & 3\\ -1 & 0 & 2\end{pmatrix}+\begin{pmatrix}-1 & 5 & -2\\ 2 & 2 & -1\end{pmatrix}=\begin{pmatrix}1 & 9 & 4\\ 0 & 2 & 3\end{pmatrix}$$

$$A-2B=\begin{pmatrix}1 & 2 & 3\\ -1 & 0 & 2\end{pmatrix}-2\begin{pmatrix}-1 & 5 & -2\\ 2 & 2 & -1\end{pmatrix}=\begin{pmatrix}3 & -8 & 7\\ -5 & -4 & 4\end{pmatrix}$$

$$A^T=\begin{pmatrix}1 & -1\\ 2 & 0\\ 3 & 2\end{pmatrix}$$

$$(A+B)^T=\begin{pmatrix}0 & 7 & 1\\ 1 & 2 & 1\end{pmatrix}^T=\begin{pmatrix}0 & 1\\ 7 & 2\\ 1 & 1\end{pmatrix}$$

2. Write down the row vectors and column vectors of matrices A, B in problem 1.

$$A=\begin{pmatrix}1 & 2 & 3\\ -1 & 0 & 2\end{pmatrix}=\begin{pmatrix}a_1\\ a_2\end{pmatrix}=\begin{pmatrix}b_1 & b_2 & b_3\end{pmatrix}$$

where $a_1=\begin{pmatrix}1 & 2 & 3\end{pmatrix}, a_2=\begin{pmatrix}-1 & 0 & 2\end{pmatrix}; b_1=\begin{pmatrix}1\\ -1\end{pmatrix}, b_2=\begin{pmatrix}2\\ 0\end{pmatrix}, b_3=\begin{pmatrix}3\\ 2\end{pmatrix}$

$$B=\begin{pmatrix}-1 & 5 & -2\\ 2 & 2 & -1\end{pmatrix}=\begin{pmatrix}a_1\\ a_2\end{pmatrix}=\begin{pmatrix}b_1 & b_2 & b_3\end{pmatrix}$$

where $a_1=\begin{pmatrix}-1 & 5 & -2\end{pmatrix}, a_2=\begin{pmatrix}2 & 2 & -1\end{pmatrix}; b_1=\begin{pmatrix}-1\\ 2\end{pmatrix}, b_2=\begin{pmatrix}5\\ 2\end{pmatrix}, b_3=\begin{pmatrix}-2\\ -1\end{pmatrix}$

3. Let $A=\begin{pmatrix}1 & 2\\ 3 & -1\end{pmatrix}$ and $B=\begin{pmatrix}2 & 0\\ 1 & 1\end{pmatrix}$. Find AB and BA.

$$AB=\begin{pmatrix}1 & 2\\ 3 & -1\end{pmatrix}\begin{pmatrix}2 & 0\\ 1 & 1\end{pmatrix}=\begin{pmatrix}4 & 2\\ 5 & -1\end{pmatrix}$$

$$BA=\begin{pmatrix}2 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & -1\end{pmatrix}=\begin{pmatrix}2 & 4\\ 4 & 1\end{pmatrix}$$

4. Let $A=\begin{pmatrix}2 & -1 & 1\\ 3 & 0 & -1\end{pmatrix}$, $B=\begin{pmatrix}1 & 1 & 0\\ 2 & 1 & -1\\ 3 & 1 & 5\end{pmatrix}$ and $C=\begin{pmatrix}1 & 2\\ 3 & 1\\ -1 & 4\end{pmatrix}$. Find $(AB)C$ and $A(BC)$.

$$(AB)C=\left(\begin{pmatrix}2 & -1 & 1\\ 3 & 0 & -1\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\ 2 & 1 & -1\\ 3 & 1 & 5\end{pmatrix}\right)\begin{pmatrix}1 & 2\\ 3 & 1\\ -1 & 4\end{pmatrix}=\begin{pmatrix}3 & 2 & 6\\ 0 & 2 & -5\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 1\\ -1 & 4\end{pmatrix}=\begin{pmatrix}3 & 32\\ 11 & -18\end{pmatrix}$$

$$A(BC)=\begin{pmatrix}2 & -1 & 1\\ 3 & 0 & -1\end{pmatrix}\left(\begin{pmatrix}1 & 1 & 0\\ 2 & 1 & -1\\ 3 & 1 & 5\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 1\\ -1 & 4\end{pmatrix}\right)=\begin{pmatrix}2 & -1 & 1\\ 3 & 0 & -1\end{pmatrix}\begin{pmatrix}4 & 3\\ 6 & 1\\ 1 & 27\end{pmatrix}=\begin{pmatrix}3 & 32\\ 11 & -18\end{pmatrix}$$

5. Let A be an $n\times n$ matrix. Define the trace of A to be the sum of the diagonal elements. Thus, if $A=(a_{ij})$, then

$$\mathrm{tr}(A)=\sum_{i=1}^{n}a_{ii}$$

Compute the trace of the matrix $A=\begin{pmatrix}1 & 7 & 3\\ -1 & 5 & 2\\ 2 & 3 & -4\end{pmatrix}$.

$\mathrm{trace}(A)=1+5+(-4)=2$.

6. Find the rank of the following matrices:

(1) $\begin{pmatrix}2 & 1 & 3\\ 7 & 2 & 0\end{pmatrix}$ (2) $\begin{pmatrix}1 & 2 & -3\\ -1 & -2 & 3\\ 4 & 8 & -12\\ 0 & 0 & 0\end{pmatrix}$ (3) $\begin{pmatrix}-1 & 0 & 1\\ 0 & 2 & 3\\ 0 & 0 & 7\end{pmatrix}$

(1) $\begin{pmatrix}2 & 1 & 3\\ 7 & 2 & 0\end{pmatrix}\sim\begin{pmatrix}2 & 1 & 3\\ 0 & -\frac{3}{2} & -\frac{21}{2}\end{pmatrix}$, $\therefore \mathrm{rank}(\begin{pmatrix}2 & 1 & 3\\ 7 & 2 & 0\end{pmatrix})=2$

(2) $\begin{pmatrix}1 & 2 & -3\\ -1 & -2 & 3\\ 4 & 8 & -12\\ 0 & 0 & 0\end{pmatrix}\sim\begin{pmatrix}1 & 2 & -3\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$, $\therefore \mathrm{rank}(\begin{pmatrix}1 & 2 & -3\\ -1 & -2 & 3\\ 4 & 8 & -12\\ 0 & 0 & 0\end{pmatrix})=1$

(3) $\begin{pmatrix}-1 & 0 & 1\\ 0 & 2 & 3\\ 0 & 0 & 7\end{pmatrix}$, $\therefore \mathrm{rank}(\begin{pmatrix}-1 & 0 & 1\\ 0 & 2 & 3\\ 0 & 0 & 7\end{pmatrix})=3$

7. Computer the following determinants:

(1) $\begin{vmatrix} 2 & 1 & 2 \\ 0 & 3 & -1 \\ 4 & 1 & 1 \end{vmatrix}$ (2) $\begin{vmatrix} 2 & 4 & 3 \\ -1 & 3 & 0 \\ 0 & 2 & 1 \end{vmatrix}$ (3) $\begin{vmatrix} 1 & 1 & -2 & 4 \\ 0 & 1 & 1 & 3 \\ 2 & -1 & 1 & 0 \\ 3 & 1 & 2 & 5 \end{vmatrix}$ (4) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

(1) $\begin{vmatrix} 2 & 1 & 2 \\ 0 & 3 & -1 \\ 4 & 1 & 1 \end{vmatrix} = 6+0-4-24+2-0=-20$

(2) $\begin{vmatrix} 2 & 4 & 3 \\ -1 & 3 & 0 \\ 0 & 2 & 1 \end{vmatrix} = 6-6+0-0-0+4=4$

(3) $\begin{vmatrix} 1 & 1 & -2 & 4 \\ 0 & 1 & 1 & 3 \\ 2 & -1 & 1 & 0 \\ 3 & 1 & 2 & 5 \end{vmatrix} = 1\times\begin{vmatrix} 1 & 1 & 3 \\ -1 & 1 & 0 \\ 1 & 2 & 5 \end{vmatrix} + 2\times\begin{vmatrix} 1 & -2 & 4 \\ 1 & 1 & 3 \\ 1 & 2 & 5 \end{vmatrix} - 3\times\begin{vmatrix} 1 & -2 & 4 \\ 1 & 1 & 3 \\ -1 & 1 & 0 \end{vmatrix} = 1+14-33=-18$

(4) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$

$= \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} = (b-a)(c-a)(c-b) = (a-b)(b-c)(c-a)$

8. Solve the following systems of linear equations:

(1) $\begin{cases} 2x-y+z=1 \\ x+3y-2z=0 \\ 4x-3y+z=2 \end{cases}$ (2) $\begin{cases} 4x+y+z+w=1 \\ x-y+2z-3w=0 \\ 2x+y+3z+5w=0 \\ x+y-z-w=2 \end{cases}$

(1)

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 1 & 3 & -2 & 0 \\ 4 & -3 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 2 & -1 & 1 & 1 \\ 4 & -3 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & -7 & 5 & 1 \\ 0 & -15 & 9 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & 1 & -\frac{5}{7} & -\frac{1}{7} \\ 0 & -15 & 9 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{7} & \frac{3}{7} \\ 0 & 1 & -\frac{5}{7} & -\frac{1}{7} \\ 0 & 0 & -\frac{12}{7} & -\frac{1}{7} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{7} & \frac{5}{7} \\ 0 & 1 & -\frac{5}{7} & -\frac{1}{7} \\ 0 & 0 & 1 & \frac{1}{12} \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{5}{12} \\ 0 & 1 & 0 & -\frac{1}{12} \\ 0 & 0 & 1 & \frac{1}{12} \end{array}\right]$$

$\therefore x=\frac{5}{12}, y=-\frac{1}{12}, z=\frac{1}{12}$

(2)

$$\left[\begin{array}{cccc|c} 4 & 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -3 & 0 \\ 2 & 1 & 3 & 5 & 0 \\ 1 & 1 & -1 & -1 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & 2 & -3 & 0 \\ 4 & 1 & 1 & 1 & 1 \\ 2 & 1 & 3 & 5 & 0 \\ 1 & 1 & -1 & -1 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & 2 & -3 & 0 \\ 0 & 5 & -7 & 13 & 1 \\ 0 & 3 & -1 & 11 & 0 \\ 0 & 2 & -3 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & 2 & -3 & 0 \\ 0 & 1 & -\tfrac{7}{5} & \tfrac{13}{5} & \tfrac{1}{5} \\ 0 & 3 & -1 & 11 & 0 \\ 0 & 2 & -3 & 2 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & \tfrac{3}{5} & -\tfrac{2}{5} & \tfrac{1}{5} \\ 0 & 1 & -\tfrac{7}{5} & \tfrac{13}{5} & \tfrac{1}{5} \\ 0 & 0 & \tfrac{16}{5} & \tfrac{16}{5} & -\tfrac{3}{5} \\ 0 & 0 & -\tfrac{1}{5} & -\tfrac{16}{5} & \tfrac{8}{5} \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & \tfrac{3}{5} & \tfrac{2}{5} & \tfrac{1}{5} \\ 0 & 1 & -\tfrac{7}{5} & \tfrac{13}{5} & \tfrac{1}{5} \\ 0 & 0 & 1 & 1 & -\tfrac{3}{16} \\ 0 & 0 & -\tfrac{1}{5} & -\tfrac{16}{5} & \tfrac{8}{5} \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & \tfrac{5}{16} \\ 0 & 1 & 0 & 4 & -\tfrac{1}{16} \\ 0 & 0 & 1 & 1 & -\tfrac{3}{16} \\ 0 & 0 & 0 & -3 & \tfrac{25}{16} \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & \tfrac{5}{16} \\ 0 & 1 & 0 & 4 & -\tfrac{1}{16} \\ 0 & 0 & 1 & 1 & -\tfrac{3}{16} \\ 0 & 0 & 0 & 1 & -\tfrac{25}{48} \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -\tfrac{5}{24} \\ 0 & 1 & 0 & 0 & \tfrac{97}{48} \\ 0 & 0 & 1 & 0 & \tfrac{1}{3} \\ 0 & 0 & 0 & 1 & -\tfrac{25}{48} \end{array}\right]$$

$\therefore x = -\tfrac{5}{24}, y = \tfrac{97}{48}, z = \tfrac{1}{3}, w = -\tfrac{25}{48}$

9. Find the inverse of the following matrices:

(1) $\begin{pmatrix} 3 & -1 \\ 1 & 4 \end{pmatrix}$ (2) $\begin{pmatrix} 1 & 3 & -2 \\ 2 & 5 & -3 \\ -3 & 2 & -4 \end{pmatrix}$ (3) $\begin{pmatrix} 1 & -1 & 3 & 1 \\ 1 & 0 & 1 & 2 \\ 3 & 0 & 2 & -1 \\ 4 & 2 & 1 & 16 \end{pmatrix}$

(1)

$$\left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 3 & -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 0 & -13 & 1 & -3 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 0 & 1 & -\tfrac{1}{13} & \tfrac{3}{13} \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & \tfrac{4}{13} & \tfrac{1}{13} \\ 0 & 1 & -\tfrac{1}{13} & \tfrac{3}{13} \end{array}\right]$$

$$\therefore \begin{bmatrix} 3 & -1 \\ 1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} \tfrac{4}{13} & \tfrac{1}{13} \\ -\tfrac{1}{13} & \tfrac{3}{13} \end{bmatrix}$$

(2)

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 2 & 5 & -3 & 0 & 1 & 0 \\ -3 & 2 & -4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 \\ 0 & 11 & -10 & 3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & -1 & 0 \\ 0 & 11 & -10 & 3 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -5 & 3 & 0 \\ 0 & 1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -19 & 11 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 14 & -8 & -1 \\ 0 & 1 & 0 & -17 & 10 & 1 \\ 0 & 0 & 1 & -19 & 11 & 1 \end{array}\right]$$

$$\therefore \begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -3 \\ -3 & 2 & -4 \end{bmatrix}^{-1} = \begin{bmatrix} 14 & -8 & -1 \\ -17 & 10 & 1 \\ -19 & 11 & 1 \end{bmatrix}$$

(3)

$$\left[\begin{array}{cccc|cccc}1 & -1 & 3 & 1 & 1 & 0 & 0 & 0\\ 1 & 0 & 1 & 2 & 0 & 1 & 0 & 0\\ 3 & 0 & 2 & -1 & 0 & 0 & 1 & 0\\ 4 & 2 & 1 & 16 & 0 & 0 & 0 & 1\end{array}\right] \sim \left[\begin{array}{cccc|cccc}1 & -1 & 3 & 1 & 1 & 0 & 0 & 0\\ 0 & 1 & -2 & 1 & -1 & 1 & 0 & 0\\ 0 & 3 & -7 & -4 & -3 & 0 & 1 & 0\\ 0 & 6 & -11 & 12 & -4 & 0 & 0 & 1\end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc}1 & 0 & 1 & 2 & 0 & 1 & 0 & 0\\ 0 & 1 & -2 & 1 & -1 & 1 & 0 & 0\\ 0 & 0 & -1 & -7 & 0 & -3 & 1 & 0\\ 0 & 0 & 1 & 6 & 2 & -6 & 0 & 1\end{array}\right] \sim \left[\begin{array}{cccc|cccc}1 & 0 & 1 & 2 & 0 & 1 & 0 & 0\\ 0 & 1 & -2 & 1 & -1 & 1 & 0 & 0\\ 0 & 0 & 1 & 7 & 0 & 3 & -1 & 0\\ 0 & 0 & 1 & 6 & 2 & -6 & 0 & 1\end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc}1 & 0 & 0 & -5 & 0 & -2 & 1 & 0\\ 0 & 1 & 0 & 15 & -1 & 7 & -2 & 0\\ 0 & 0 & 1 & 7 & 0 & 3 & -1 & 0\\ 0 & 0 & 0 & -1 & 2 & -9 & 1 & 1\end{array}\right] \sim \left[\begin{array}{cccc|cccc}1 & 0 & 0 & -5 & 0 & -2 & 1 & 0\\ 0 & 1 & 0 & 15 & -1 & 7 & -2 & 0\\ 0 & 0 & 1 & 7 & 0 & 3 & -1 & 0\\ 0 & 0 & 0 & 1 & -2 & 9 & -1 & -1\end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc}1 & 0 & 0 & 0 & -10 & 43 & -4 & -5\\ 0 & 1 & 0 & 0 & 29 & -128 & 13 & 15\\ 0 & 0 & 1 & 0 & 14 & -60 & 6 & 7\\ 0 & 0 & 0 & 1 & -2 & 9 & -1 & -1\end{array}\right]$$

$$\therefore \begin{bmatrix}1 & -1 & 3 & 1\\ 1 & 0 & 1 & 2\\ 3 & 0 & 2 & -1\\ 4 & 2 & 1 & 16\end{bmatrix}^{-1} = \begin{bmatrix}-10 & 43 & -4 & -5\\ 29 & -128 & 13 & 15\\ 14 & -60 & 6 & 7\\ -2 & 9 & -1 & -1\end{bmatrix}$$

10. Find the relation between b_1, b_2, and b_3 such that the system $\begin{cases}x_1 + 5x_2 + 2x_3 = b_1\\ 2x_1 + x_2 + x_3 = b_2\\ x_1 + 2x_2 + x_3 = b_3\end{cases}$ has a solution.

$$\left[\begin{array}{ccc|c}1 & 5 & 2 & b_1\\ 2 & 1 & 1 & b_2\\ 1 & 2 & 1 & b_3\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 5 & 2 & b_1\\ 0 & -9 & -3 & b_2 - 2b_1\\ 0 & -3 & -1 & b_3 - b_1\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 5 & 2 & b_1\\ 0 & 1 & \frac{1}{3} & \dfrac{2b_1 - b_2}{9}\\ 0 & -3 & -1 & b_3 - b_1\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 0 & \dfrac{1}{3} & \dfrac{14b_1 - 10b_2}{9}\\ 0 & 1 & \dfrac{1}{3} & \dfrac{2b_1 - b_2}{9}\\ 0 & 0 & 0 & \dfrac{-b_1 - b_2 + 3b_3}{3}\end{array}\right]$$

(1) When $b_1 + b_2 = 3b_3$, the system has a multiple solution.

(2) When $b_1 + b_2 \neq 3b_3$, the system has no solution.