The Linear Programming-the Simplex Algorithm

Linear Programming (LP)(線性規劃) is a tool for solving optimization problems. In 1947, **George Dantzig** developed an efficient method, the **simplex algorithm**(單形法), for solving linear programming problems. Since the development of the simplex algorithm, LP has been used to solve optimization problems in industries are diverse as banking, education, forestry, petroleum, and trucking. In a survey of Fortune 500 firms, 85% of the respondents said they had used LP. As a measure of the importance of LP in OR, approximately 70% of this book will be devoted to LP and related optimization techniques.

We devote to a discussion of the **simplex algorithm**, which is used to solve even very large LPs. In many industrial applications, the simplex algorithm is used to solve LPs with thousands of constraints and variables. We should explain how the simplex algorithm can be used to find optimal solutions to LPs., and detail how two state-of-the-art computer packages (LINDO) can be used to solve LPs.

Type 1: Graphical Solution(圖解法)

Example:

The WYNDOR GLASS CO. produces high-quality glass products, including windows and glass doors. It has three plants. Aluminum frames and hardware are made in Plant 1, wood frame are madder in Plant 2, and Plant 3 produces the glass and assembles the products.

Because of declining earnings, top management has decided to revamp the company's product line. Unprofitable products are being discontinued, releasing production capacity to launch two new products having large sales potential:

Product 1: An 8-foot glass door with aluminum framing.

Product 2: A 4×6 foot double-hung wood-framed window.

Table Data for the Wyndor Glass Co. problem

	Production Time per Batch, Hours		
	Product		Production Time Available per Week, Hours
Plant	1	2	
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	\$3000	\$5000	

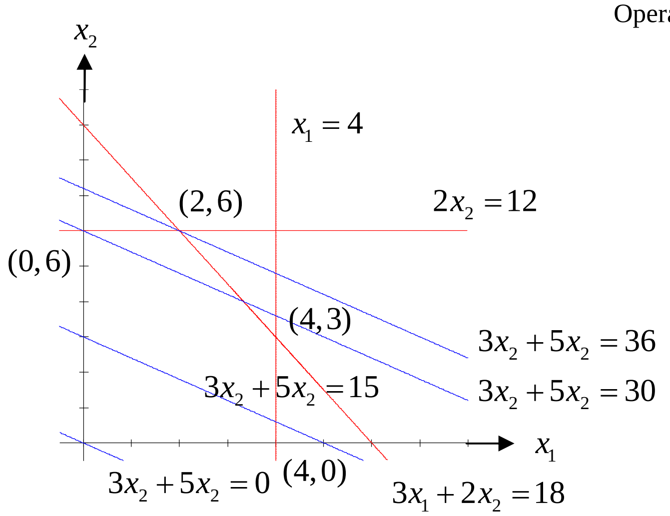
Solution: We define

x_1 = number of batches of product 1 produced per week

x_2 = number of batches of product 2 produced per week

An LP is

$$\begin{aligned} \max \quad & z = 3x_1 + 5x_2 \\ \text{s.t.} \quad & x_1 \le 4 \\ & 2x_2 \le 12 \\ & 3x_1 + 2x_2 \le 18 \\ & x_1, x_2 \ge 0 \end{aligned}$$



The solution indicates that the Wyndor Glass Co. should produce products 1 and 2 at the rate of 2 batches per week and 6 batches per week, respectively, with a resulting total profit of $36000 per week.

Terminology for Solutions of the Model

1. A **feasible solution(可行解)** is a solution for which all the constraints are satisfied.
2. An **infeasible solution(不可行解)** is a solution for which at least one constraint is violated.
3. The **feasible region(可行解區域)** is the collection of all feasible solutions.
4. An **optimal solution(最佳解)** is a feasible solution that has the most favorable value of the objective function.
5. A **corner-point feasible (CPF; basic feasible)(基本可行解)** solution is a solution that lies at a corner of the feasible region.

Relationship between optimal solutions and CPF solutions

Consider any linear programming problem with feasible solutions and a bounded feasible region. The problem must posses CPF solutions and **at least** one optimal solution. Furthermore, the **best CPF solution must be an optimal solution**. Thus, if a problem has exactly one optimal solution, it must be a CPF solution. If the problem has multiple optimal solutions, at least two must be CPF solution.

The solution for Linear Programming:

1. Uniquely optimal solution(唯一解)
2. Multiple optimal solutions(無限多組解)
3. Unbounded(無界)
4. No feasible solution(無可行解)

Assumptions of Linear Programming:

1. **Proportionality(可比例性)** —非線性規劃(Nonlinear Programming)
2. **Additivity(可加性)** — 非線性規劃(Nonlinear Programming)
3. **Divisibility(可分性)** — 整數規劃(Integer Programming)
4. **Certainty(確定性)** — 隨機模式(Stochastic Model)

Type 2: Simplex method(單形法)

Convert an LP to Standard Form

We have seen that an LP can have both **equality and inequality constraints**. It also can have **variables** that are require to be **nonnegative** as well as those allowed to be **unrestricted in sign**. Before the simplex algorithm can be used to solve an LP, the LP must be converted into an equivalent problem in which **all constraints are equation** and **all variables are nonnegative**. An LP in this form is said to be in standard form.

Standard Form

$$\begin{aligned} \text{max(or min)} \quad & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ & \quad \vdots \qquad \vdots \qquad\qquad \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\ & x_i \geq 0 (i = 1, 2, \cdots, n) \end{aligned}$$

Example Leather Limited

Leather Limited manufacturers two types of belts(腰帶): the deluxe(高級的)model and the regular model. Each type requires 1 sq yd of leather(皮革). A regular belt requires 1 hour of skilled labor, and a deluxe belt requires 2 hours. Each week, 40 sq yd of leather and 60 hours of skilled labor are available. Each regular belt contributes $3 to profit and each deluxe belt, $4. If we define

$x_1 =$ number of deluxe belts produced weekly

$x_2 =$ number of regular belts produced weekly

An LP is

$$\begin{aligned} \max \quad & z = 4x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 40 \\ & 2x_1 + x_2 \leq 60 \\ & x_1, x_2 \geq 0 \end{aligned}$$

A standard form of LP is

$$\begin{aligned} \max \quad & z = 4x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 + s_1 \quad = 40 \\ & 2x_1 + x_2 \quad + s_2 = 60 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{aligned}$$

Example

An LP is

$$\begin{aligned} \min \quad & z = 50x_1 + 20x_2 + 30x_3 + 80x_4 \\ \text{s.t.} \quad & 400x_1 + 200x_2 + 150x_3 + 500x_4 \geq 500 \\ & 3x_1 + 2x_2 \geq 6 \\ & 2x_1 + 2x_2 + 4x_3 + 4_4 \geq 10 \\ & 2x_1 + 4x_2 + x_3 + 5_4 \geq 8 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

A standard form of LP is

$$\begin{array}{ll} \min & z = 50x_1 + 20x_2 + 30x_3 + 80x_4 \\ \text{s.t.} & 400x_1 + 200x_2 + 150x_3 + 500x_4 - e_1 = 500 \\ & 3x_1 + 2x_2 - e_2 = 6 \\ & 2x_1 + 2x_2 + 4x_3 + 4_4 - e_3 = 10 \\ & 2x_1 + 4x_2 + x_3 + 5_4 - e_4 = 8 \\ & x_1, x_2, x_3, x_4, e_1, e_2, e_3, e_4 \geq 0 \end{array}$$

Preview of the Simplex Algorithm

$$\begin{array}{ll} \max & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{s.t.} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ & \quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\ & x_i \geq 0 (i = 1, 2, \cdots, n) \end{array}$$

Matrix Form:

$$\begin{array}{ll} \max & z = C^T X \\ \text{s.t.} & AX = b \\ & X \geq 0 \end{array}$$

where

$$C^T = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Basic Properties of Liner Programming

Consider the system of equalities

$$AX = b$$

where X is an n-vector, b an m-vector, and A is an $m \times n$ matrix. Suppose that from the n columns of A we select a set of m linearly independent columns (such a set exists if the rank of A is m). For notational simplicity assume that we select the first m columns of A and denote the $m \times m$ matrix determined by these columns by B. The matrix B is then nonsingular and we may **uniquely** solve the equation

$$BX_B = b$$

for the m vector X_B. By putting $X = \left(X_B, \vec{0} \right)$ (that is, setting the first m components of X equal to those of X_B and the remaining components equal to zero), we obtain a solution to $AX = b$. This leads to the following definition.

Definition of Basic Variable

Given the set of m simultaneous linear equations in n unknowns $AX = b$, let B be any nonsingular $m \times m$ submatrix made up of columns of A. Then, if all $m - n$ components of X not associated with columns of B are set equal to zero, the solution to the resulting set of equations is said to be a **basic solution**(**基解**) to (1) with respect to the basis B. The components of X associated with columns of B are called **basic variables**(**BV**, **基變數**), otherwise are called **nonbasic variables(NBV, 非基變數)**.

To find a basic solution to $AX = b$, we choose a set of $n - m$ variables (the **nonbasic variables**, or **NBV(非基變數)**) and set each of these variables equal to 0. Then we solve for the values of the remaining m variables (the basic variables, BV) that satisfy $AX = b$.

Example:

$$x_1 + x_2 \quad\quad = 3$$
$$-x_2 + x_3 = -1$$

If $\text{NBV} = \{x_3\}$, then $\text{BV} = \{x_1, x_2\}$. We obtain the values of the basic variables by setting $x_3 = 0$ and solving

$$x_1 + x_2 = 3$$
$$-x_2 = -1$$

We find that $x_1 = 2, x_2 = 1$.

Definition of Feasible and Basic Feasible Solution

A vector X satisfying $AX = b$ and $X \geq 0$ is said to the **feasible solution(可行解)** for these constraints.

A feasible solution to the constraints $AX = b$ and $X \geq 0$ that is also basic solution is said to be a **basic feasible solution(bfs; 基本可行解)**.

Corresponding to a linear program in standard form

max(or min)	$z = C^T X$
s.t.	$AX = b$
	$X \geq 0$

a feasible solution to the constraints that achieves the maximum (or minimum) value of the objective function subject to those constraints is said to be an **optimal feasible solution**. If this solution is basic, it is an **optimal basic feasible solution**.

Theorem (Fundamental Theorem of Linear Programming)

Given a linear programming in standard form where A is an $m \times n$ matrix of rank m.

(1) If there is a feasible solution, there is a basic feasible solution.

(2) If there is an optimal feasible solution, there is an optimal basic feasible solution.

Proof: (See Appendix 1)

Remark:

This theorem reduces the task of solving a linear programming problem to that of searching over basic feasible solutions. Since for a problem having n variables and m constraints there are at most

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$$C_m^n = \frac{n!}{m!(n-m)!}$$

basic solutions, there are only a finite number of possibilities.

Definition of Convex

A set C in E^n is said to be **convex(凸集合)** if for every $x_1, x_2 \in C$ and every real number α, $0 < \alpha < 1$, the point $\alpha x_1 + (1-\alpha)x_2 \in C$.

Definition of Half Space

Let a be a nonzero vector in E^n and let c a real number. Corresponding to the **hyperplane(半平面)** $H = \{x : a^T x = c\}$ are the positive and negative **closed half spaces**

$$H_+ = \{x : a^T x \geq c\}$$
$$H_- = \{x : a^T x \leq c\}$$

and the positive and negative **open half spaces**

$$H_+ = \{x : a^T x > c\}$$
$$H_- = \{x : a^T x < c\}$$

Definition of Polytope

A set which can be expressed as the intersection of a finite number of closed half spaces is said to be a **convex polytope**.

Definition of Extreme Point

A point x in a convex set C is said to be an **extreme point(極點)** of C if there are no two distinct points x_1 and x_2 in C such that $x = \alpha x_1 + (1-\alpha)x_2$ for some α, $0 < \alpha < 1$.

Theorem (Equivalence of Extreme Points and Basic Feasible Solution)

Let A be an $m \times n$ matrix of rank m and b an m-vector. Let K be the convex polytope consisting of all n-vectors X satisfying

$$AX = b$$
$$X \geq 0$$

A vector X is an **extreme point** of K if and only if X is a **basic feasible solution** to $AX = b$ and $X \geq 0$.

Proof: (See the Appendix 2)

Corollary:

If the convex K corresponding to $AX = b$ and $X \geq 0$ is **nonempty**, it has at least one extreme point.

Corollary:

If there is a finite optimal solution to a linear programming problem, there is a finite optimal solution which is an extreme point of the constraint set.

Corollary:

The constraint set K corresponding to $AX = b$ and $X \geq 0$ possesses at most a finite

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number of extreme points.

Proof:

There are obviously only a finite of basic solutions obtained by selecting m basis vectors from the n columns of A. The extreme points of K are a subset of these basic solutions.

Example:

$$\begin{aligned} \max \quad & z = 4x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 40 \\ & 2x_1 + x_2 \le 60 \\ & x_1, x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} \max \quad & z = 4x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 + s_1 = 40 \\ & 2x_1 + x_2 + s_2 = 60 \\ & x_1, x_2, s_1, s_2 \ge 0 \end{aligned}$$

Basic Variables	Nonbasic Variables	Basic Feasible Solution	Corresponds to Corner Point
x_1, x_2	s_1, s_2	$x_1 = x_2 = 20$	E
x_1, s_1	x_2, s_2	$x_1 = 30, s_1 = 10$	C
x_1, s_2	x_2, s_1	$x_1 = 40, s_2 = -20$	Not a bfs
x_2, s_1	x_1, s_2	$x_2 = 60, s_1 = -20$	Not a bfs
x_2, s_2	x_1, s_1	$x_2 = 40, s_2 = 20$	B
s_1, s_2	x_1, x_2	$s_1 = 40, s_2 = 60$	F

Adjacent Basic Feasible Solutions

For any LP with m constraints, two basic feasible solutions are said to be **adjacent(相鄰)** if their sets of basic variables have $m-1$ basic variables in common.

Example:

$$\begin{aligned} \max \quad & z = 4x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 + s_1 = 40 \\ & 2x_1 + x_2 + s_2 = 60 \\ & x_1, x_2, s_1, s_2 \ge 0 \end{aligned}$$

The basic feasible solutions (0,0,40,60), (30,0,10,0) are adjacent.

The basic feasible solutions (30,0,10,0), (20,20,0,0) are adjacent.

The basic feasible solutions (20,20,0,0), (0,40,20,0) are adjacent.

The basic feasible solutions (0,0,40,60), (20,20,0,0) are not adjacent.

The Simplex Algorithm

Step 1: Convert the LP to standard form.

$$\begin{aligned} \max \quad & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ & \quad \vdots \qquad \vdots \qquad\qquad \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\ & x_i \ge 0 (i = 1, 2, \cdots, n) \end{aligned}$$

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Step 2: Obtain a bfs (if possible) from the standard form.

Step 3: Determine whether the current bfs is optimal.

If the $\bar{c}_j \geq 0$, then current basic feasible solution is optimal, stop.

Step 4: If the current bfs is not optimal, then determine which nonbasic variable should become a basic variable and which basic variable should become a nonbasic variable to find a new bfs with a better objective function value.

Select x_q such that $\bar{c}_q = \min\{\bar{c}_j \mid \bar{c}_j < 0, x_j$ is a nonbasic variable$\}$ to determine which nonbasic variable is to become basic.

Calculuate the ratio $\dfrac{\bar{b}_i}{\bar{a}_{ij}}$ for $\bar{a}_{ij} > 0$, $i = 1, 2, \cdots, m$. If no $\bar{a}_{ij} > 0$, stop; the problem is unbounded. Otherwise, select p as the index i corresponding to the minimum ratio.

Step 5: Use EROs to find the new bfs with the better objective function value. Go back to step 3.

Pivot on the pqth element, updating all rows including the last.

Example: max $z = 60x_1 + 30x_2 + 20x_3$

s.t. $8x_1 + 6x_2 + x_3 \leq 48$

$4x_1 + 2x_2 + 1.5x_3 \leq 20$

$2x_1 + 1.5x_2 + 0.5x_3 \leq 8$

$x_2 \leq 5$

$x_1, x_2, x_3 \geq 0$

Convert the standard form:

max $z - 60x_1 - 30x_2 - 20x_3 = 0$

s.t. $8x_1 + 6x_2 + x_3 + s_1 = 48$

$4x_1 + 2x_2 + 1.5x_3 + s_2 = 20$

$2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8$

$x_2 + s_4 = 5$

$x_1, x_2, x_3, s_1, s_2, s_3, s_4 \geq 0$

Use the Tabular Form:

variable	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	Ratio
z	-60	-30	-20	0	0	0	0	0	
s_1	8	6	1	1	0	0	0	48	48/8=6
s_2	4	2	1.5	0	1	0	0	20	20/4=5
s_3	2	1.5	0.5	0	0	1	0	8	4/2=4
s_4	0	1	0	0	0	0	1	5	*

Since $\min\{-60, -30, -20\} = -60$, then x_1 enter the basic variable.

Since $\min\{6, 5, 4, *\} = 4$, then s_3 leave the basic variable

variable	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	Ratio
z	0	15	-5	0	0	30	0	240	
s_1	0	0	-1	1	0	-4	0	16	*
s_2	0	-1	0.5	0	1	-2	0	4	4/0.5=8
x_1	1	0.75	0.25	0	0	0.5	0	4	4/0.25=16
s_4	0	1	0	0	0	0	1	5	*

Since $\min\{-5\}=-5$, then x_3 enter the basic variable.

Since $\min\{*,8,16,*\}=8$, then s_2 leave the basic variable

variable	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	Ratio
z	0	5	0	0	10	10	0	280	
s_1	0	-2	0	1	2	-8	0	24	
x_3	0	-2	1	0	2	-4	0	8	
x_1	1	1.25	0	0	-0.5	1.5	0	2	
s_4	0	1	0	0	0	0	1	5	

Hence, $x_1=2, x_2=0, x_3=8, x_4=0, z=280$.

Example: (Using the Simplex Algorithm to Solve **Minimization** Problems)

min $z=2x_1-3x_2$

s.t. $x_1+x_2\le 4$

$x_1-x_2\le 6$

$x_1,x_2\ge 0$

Convert the standard form:

max $-z+2x_1-3x_2=0$

s.t. $x_1+x_2+s_1=4$

$x_1-x_2+s_2=6$

$x_1,x_2,s_1,s_2\ge 0$

Use the Tabular Form:

variable	x_1	x_2	s_1	s_2	RHS	Ratio
$-z$	2	-3	0	0	0	
s_1	1	1	1	0	4	4/1=4
s_2	1	-1	0	1	6	*

Since $\min\{-3\}=-3$, then x_2 enter the basic variable.

Since $\min\{4,*\}=4$, then s_1 leave the basic variable

variable	x_1	x_2	s_1	s_2	RHS	Ratio
$-z$	5	0	3	0	12	
x_2	1	1	1	0	4	
s_2	2	0	1	1	10	

Hence, $x_1=0, x_2=4, z=-12$.

Example: (**Alternative Optimal Solutions**)

max $z = 60x_1 + 35x_2 + 20x_3$

s.t. $8x_1 + 6x_2 + x_3 \le 48$

$4x_1 + 2x_2 + 1.5x_3 \le 20$

$2x_1 + 1.5x_2 + 0.5x_3 \le 8$

$x_2 \le 5$

$x_1, x_2, x_3 \ge 0$

Convert the standard form:

max $z = 60x_1 + 35x_2 + 20x_3$

s.t. $8x_1 + 6x_2 + x_3 + s_1 = 48$

$4x_1 + 2x_2 + 1.5x_3 + s_2 = 20$

$2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8$

$x_2 + s_4 = 5$

$x_1, x_2, x_3, s_1, s_2, s_3, s_4 \ge 0$

Use the Tabular Form:

variable	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	Ratio
z	-60	-35	-20	0	0	0	0	0	
s_1	8	6	1	1	0	0	0	48	48/8=6
s_2	4	2	1.5	0	1	0	0	20	20/4=5
s_3	2	1.5	0.5	0	0	1	0	8	4/2=4
s_4	0	1	0	0	0	0	1	5	*

Since $\min\{-60, -35, -20\} = -60$, then x_1 enter the basic variable.

Since $\min\{6, 5, 4, *\} = 4$, then s_3 leave the basic variable

variable	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	Ratio
z	0	10	-5	0	0	30	0	240	
s_1	0	0	-1	1	0	-4	0	16	*
s_2	0	-1	0.5	0	1	-2	0	4	4/0.5=8
x_1	1	0.75	0.25	0	0	0.5	0	4	4/0.25=16
s_4	0	1	0	0	0	0	1	5	*

Since $\min\{-5\} = -5$, then x_3 enter the basic variable.

Since $\min\{*, 8, 16, *\} = 8$, then s_2 leave the basic variable.

variable	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	Ratio
z	0	0	0	0	10	10	0	280	
s_1	0	-2	0	1	2	-8	0	24	
x_3	0	-2	1	0	2	-4	0	8	
x_1	1	1.25	0	0	-0.5	1.5	0	2	
s_4	0	1	0	0	0	0	1	5	

If x_2 enter the basic variable and x_1 leave the basic variable, then

variable	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	Ratio
z	0	0	0	0	10	10	0	280	
s_1	1.6	0	0	1	1.2	-5.6	0	27.2	
x_3	1.6	0	1	0	1.2	-1.6	0	11.2	
x_2	0.8	1	0	0	-0.4	1.2	0	1.6	
s_4	-0.8	0	0	0	0.4	-1.2	1	3.4	

Hence, $x_1 = 2, x_2 = 0, x_3 = 8, x_4 = 0, z = 280$, or $x_1 = 0, x_2 = 1.6, x_3 = 11.2, x_4 = 0, z = 280$.

Example: (**Unbounded LPs**)

$$\begin{aligned} \max \quad & z = 36x_1 + 30x_2 - 3x_3 - 4x_4 \\ \text{s.t.} \quad & x_1 + x_2 - x_3 \le 5 \\ & 6x_1 + 5x_2 - x_4 \le 10 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

Convert the standard form:

$$\begin{aligned} \max \quad & z - 36x_1 - 30x_2 + 3x_3 + 4x_4 = 0 \\ \text{s.t.} \quad & x_1 + x_2 - x_3 + s_1 = 5 \\ & 6x_1 + 5x_2 - x_4 + s_2 = 10 \\ & x_1, x_2, x_3, x_4, s_1, s_2 \ge 0 \end{aligned}$$

Table Form

variable	x_1	x_2	x_3	x_4	s_1	s_2	RHS	Ratio
z	-36	-30	3	4	0	0	0	
s_1	1	1	-1	0	1	0	5	5/1=5
s_2	6	5	0	-1	0	1	10	10/6

Since $\min\{-36, -30\} = -36$, then x_1 enter the basic variable.

Since $\min\{5, \frac{10}{6}\} = \frac{10}{6}$, then s_2 leave the basic variable.

variable	x_1	x_2	x_3	x_4	s_1	s_2	RHS	Ratio
z	0	0	3	-2	0	6	60	
s_1	0	$\frac{1}{6}$	-1	$\frac{1}{6}$	1	$-\frac{1}{6}$	$\frac{10}{3}$	$\frac{10}{3} / \frac{1}{6} = 20$
x_1	1	$\frac{5}{6}$	0	$-\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{5}{3}$	*

Since $\min\{-2\} = -2$, then x_4 enter the basic variable.

Since $\min\{20, *\} = 20$, then s_1 leave the basic variable.

variable	x_1	x_2	x_3	x_4	s_1	s_2	RHS	Ratio
z	0	$\frac{1}{3}$	-9	0	12	4	100	
x_4	0	1	-6	1	6	-1	20	
x_1	1	1	-1	0	1	0	5	

We can find points in the feasible region having arbitrarily large z-values.

Definition of Degeneracy

An LP is **degenerate(退化)** if it has at least one bfs in which a basic variable is equal to zero.

Example: max $z = 5x_1 + 2x_2$

s.t. $x_1 + x_2 \le 6$

$x_1 - x_2 \le 0$

$x_1, x_2 \ge 0$

Convert the standard form:

max $z - 5x_1 - 2x_2 = 0$

s.t. $x_1 + x_2 + s_1 = 6$

$x_1 - x_2 + s_2 = 0$

$x_1, x_2, s_1, s_2 \ge 0$

variable	x_1	x_2	s_1	s_2	RHS	Ratio
z	-5	-2	0	0	0	
s_1	1	1	1	0	6	6/1=6
s_2	1	-1	0	1	0	0/1=0

Since $\min\{-5,-2\} = -5$, then x_1 enter the basic variable.

Since $\min\{6,0\} = 0$, then s_2 leave the basic variable.

variable	x_1	x_2	s_1	s_2	RHS	Ratio
z	0	-7	0	5	0	
s_1	0	2	1	-1	6	6/2=3
x_1	1	-1	0	1	0	*

Since $\min\{-7\} = -7$, then x_2 enter the basic variable.

Since $\min\{3,*\} = 3$, then s_1 leave the basic variable.

variable	x_1	x_2	s_1	s_2	RHS	Ratio
z	0	0	3.5	1.5	21	
x_2	0	1	0.5	-0.5	3	
x_1	1	0	0.5	0.5	3	

Termination is not guaranteed foe degenerate problems. Consider the linear program

max $z = -\frac{3}{4}x_1 + 150x_2 - \frac{1}{50}x_3 + 6x_4$

s.t. $\frac{1}{4}x_1 - 60x_2 - \frac{1}{25}x_3 + 9x_4 \le 0$

$\frac{1}{2}x_1 - 90x_2 - \frac{1}{50}x_3 + 3x_4 \le 0$

$x_3 \le 1$

$x_1, x_2, x_3, x_4 \ge 0$

We will apply the simplex method to this problem, using the most negative reduced cost to select the entering variable, and breaking ties in the ratio test by selecting the first candidate row. If this is done, then the simplex method cycles—endlessly repeating the same sequence of bases with no improvement in the objective and without finding the optimal solution.

variable	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
$-z$	$-\frac{3}{4}$	150	$-\frac{1}{50}$	6	0	0	0	0
s_1	$\frac{1}{4}$	-60	$-\frac{1}{25}$	9	1	0	0	0
s_2	$\frac{1}{2}$	-90	$-\frac{1}{50}$	3	0	1	0	0
s_3	0	0	1	0	0	0	1	1

variable	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
$-z$	0	-30	$-\frac{7}{50}$	33	3	0	0	0
x_1	1	-240	$-\frac{4}{25}$	36	4	0	0	0
s_2	0	30	$\frac{3}{50}$	-15	-2	1	0	0
s_3	0	0	1	0	0	0	1	1

variable	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
$-z$	0	0	$-\frac{2}{25}$	18	1	1	0	0
x_1	1	0	$\frac{8}{25}$	-84	-12	8	0	0
x_2	0	1	$\frac{1}{500}$	$-\frac{1}{2}$	$-\frac{1}{15}$	$\frac{1}{30}$	0	0
s_3	0	0	1	0	0	0	1	1

variable	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
$-z$	$\frac{1}{4}$	0	0	-3	-2	3	0	0
x_3	$\frac{25}{8}$	0	1	$-\frac{525}{2}$	$-\frac{75}{2}$	25	0	0
x_2	$-\frac{1}{160}$	1	0	$\frac{1}{40}$	$\frac{1}{120}$	$-\frac{1}{60}$	0	0
s_3	$-\frac{25}{8}$	0	1	$\frac{525}{2}$	$\frac{75}{2}$	-25	1	1

variable	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
$-z$	$-\frac{1}{2}$	120	0	0	-1	1	0	0
x_3	$-\frac{125}{2}$	10500	1	0	50	-150	0	0
x_4	$-\frac{1}{4}$	40	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	0	0
s_3	$\frac{125}{2}$	-10500	0	0	-50	150	1	1

variable	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
$-z$	$-\frac{7}{4}$	330	$\frac{1}{50}$	0	0	-2	0	0
x_5	$-\frac{5}{4}$	210	$\frac{1}{50}$	0	1	-3	0	0
x_4	$\frac{1}{6}$	-30	$-\frac{1}{150}$	1	0	$\frac{1}{3}$	0	0
s_3	0	0	1	0	0	0	1	1

variable	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
$-z$	$-\frac{3}{4}$	150	$-\frac{1}{50}$	6	0	0	0	0
s_1	$\frac{1}{4}$	-60	$-\frac{1}{25}$	9	1	0	0	0
s_2	$\frac{1}{2}$	-90	$-\frac{1}{50}$	3	0	1	0	0
s_3	0	0	1	0	0	0	1	1

The final basis is the same as the initial basis, so that the simplex method has made no progress and will continue to cycle through these six bases indefinitely.

A variety of techniques have been developed that guarantee termination of the simplex method even on degenerate problems. One of these, discovered by Bland and often referred to as "**Bland's rule**," is described here.

variable	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
$-z$	$-\frac{1}{2}$	120	0	0	-1	1	0	0
x_3	$-\frac{125}{2}$	10500	1	0	50	-150	0	0
x_4	$-\frac{1}{4}$	40	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	0	0
s_3	$\frac{125}{2}$	-10500	0	0	-50	150	1	1

variable	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
$-z$	0	36	0	0	$-\frac{7}{5}$	$\frac{11}{5}$	$\frac{1}{125}$	$\frac{1}{125}$
x_3	0	0	1	0	0	0	1	1
x_4	0	-2	0	1	$\frac{2}{15}$	$-\frac{1}{15}$	$\frac{1}{250}$	$\frac{1}{250}$
x_1	1	-168	0	0	$-\frac{4}{5}$	$\frac{12}{5}$	$\frac{2}{125}$	$\frac{2}{125}$

variable	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
$-z$	0	15	0	$\frac{21}{2}$	0	$\frac{3}{2}$	$\frac{1}{20}$	$\frac{1}{20}$
x_3	0	0	1	0	0	0	1	1
s_1	0	-15	0	$\frac{15}{2}$	1	$-\frac{1}{2}$	$\frac{3}{100}$	$\frac{3}{100}$
x_1	1	-180	0	6	0	2	$\frac{2}{50}$	$\frac{2}{50}$

Note: Bland's rule can be inefficient if applied at every simplex iteration since it may select entering variables that do not greatly improve the value of the objective function.

(Reference: Robert G. Bland, New finite pivoting rules for the simplex method, Mathematics of Operations Research 2 (1997) pp. 103-107)

The Big M Method

Recall that the simplex algorithm requires a stating bfs. In all the problems we have solved so far, we found a starting bfs by using the slack variables as our basic variables. If an LP has any $\geq$ or equality constraints, however, a starting bfs may not readily apparent. When a bfs is not readily apparent, the Big M method (or the two-phase simplex) may be used to solve the problem. The Big M method first find a bfs by adding "artificial" variables to the problem. The objective function of the original LP must, of course, be modified to ensure that the artificial variables are all equal to 0 at the conclusion of the simplex algorithm.

Example: Bevco

Bevco manufactures an orange-flavored soft drink called Oranj by combining orange soda and orange juice. Each ounce of orange soda contains 0.5 oz of sugar and 1 mg of vitamin C. Each ounce of orange juice contains 0.25 oz of sugar and 3 mg of vitamin C. It costs Bevco 2 ¢ to produce an ounce of orange soda and 3 ¢ to produce an ounce of orange juice. Bevco's marketing department has decided that each 10-oz bottle of Oranj must contain at least 20 mg of vitamin C and at most 4 oz of sugar. Use linear programming to determine how Bevco can meet the marketing department's requirements at minimum cost.

Solution: Let

$x_1 =$ number of ounces of orange soda in a bottle of Oranj

$x_2 =$ number of ounces of orange juice in a bottle of Oranj

Then the appropriate LP is

$$\begin{aligned} \min\ & z = 2x_1 + 3x_2 \\ \text{s.t.}\ & \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 \le 4 \\ & x_1 + 3x_2 \ge 20 \\ & x_1 + x_2 = 10 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Convert the standard form:

$$\begin{aligned} \max\ & -z = -2x_1 - 3x_2 - Ma_2 - Ma_3 \\ \text{s.t.}\ & \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 + s_1 = 4 \\ & x_1 + 3x_2 - e_2 + a_2 = 20 \\ & x_1 + x_2 + a_3 = 10 \\ & x_1, x_2, s_1, e_2, a_2, a_3 \ge 0 \end{aligned}$$

Tabular Form:

	x_1	x_2	s_1	e_2	a_2	a_3	RHS	Ratio
$-z$	2	3	0	0	M	M	0	
s_1	½	¼	1	0	0	0	4	
a_2	1	3	0	-1	1	0	20	
a_3	1	1	0	0	0	1	10	

The initial table is

	x_1	x_2	s_1	e_2	a_2	a_3	RHS	Ratio
$-z$	$-2M+2$	$-4M+3$	0	M	0	0	$30M$	
s_1	½	¼	1	0	0	0	4	16
a_2	1	3	0	-1	1	0	20	20/3
a_3	1	1	0	0	0	1	10	10

Since $\min\{-2M+2, -4M+3\} = -4M+3$, then x_2 enter the basic variable.

Since $\min\{16, \frac{20}{3}, 10\} = \frac{20}{3}$, then a_2 leave the basic variable.

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	x_1	x_2	s_1	e_2	a_2	a_3	RHS	Ratio
$-z$	$\frac{-2M+3}{3}$	0	0	$\frac{-M+3}{3}$	$\frac{4M-3}{3}$	0	$\frac{10M+60}{3}$	
s_1	$\frac{5}{12}$	0	1	$\frac{1}{12}$	$-\frac{1}{12}$	0	$\frac{7}{3}$	28/5
x_2	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{20}{3}$	20
a_3	$\frac{2}{3}$	0	0	$\frac{1}{3}$	$-\frac{1}{3}$	1	$\frac{10}{3}$	5

Since $\min\{\frac{-2M+3}{3}, \frac{-M+3}{3}\} = \frac{-2M+3}{3}$, then x_1 enter the basic variable.

Since $\min\{\frac{28}{5}, 20, 5\} = 5$, then a_3 leave the basic variable.

	x_1	x_2	s_1	e_2	a_2	a_3	RHS	Ratio
$-z$	0	0	0	$\frac{1}{2}$	$\frac{2M-1}{2}$	$\frac{2M-3}{2}$	25	
s_1	0	0	1	$-\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{5}{8}$	$\frac{1}{4}$	
x_2	0	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	5	
x_1	1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	5	

Hence, $x_1 = 5, x_2 = 5, z = 25$

The Two-Phase Simplex Method

Case 1:

Example: min $z = 2x_1 + 3x_2$

s.t. $\frac{1}{2}x_1 + \frac{1}{4}x_2 \le 4$

$x_1 + 3x_2 \ge 36$

$x_1 + x_2 = 10$

$x_1, x_2 \ge 0$

Phase I problem:

min $w = a_2 + a_3$

s.t. $\frac{1}{2}x_1 + \frac{1}{4}x_2 + s_1 = 4$

$x_1 + 3x_2 - e_2 + a_2 = 36$

$x_1 + x_2 + a_3 = 10$

$x_1, x_2, s_1, e_2, a_2, a_3 \ge 0$

	x_1	x_2	s_1	e_2	a_2	a_3	RHS	Ratio
$-w$	0	0	0	0	1	1	0	
s_1	$\frac{1}{2}$	$\frac{1}{4}$	1	0	0	0	4	
a_2	1	3	0	-1	1	0	36	
a_3	1	1	0	0	0	1	10	

The initial table is

	x_1	x_2	s_1	e_2	a_2	a_3	RHS	Ratio
$-w$	-2	-4	0	1	0	0	-46	
s_1	$\frac{1}{2}$	$\frac{1}{4}$	1	0	0	0	4	16
a_2	1	3	0	-1	1	0	36	12
a_3	1	1	0	0	0	1	10	10

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Since $\min\{-2,-4\}=-4$, then x_2 enter the basic variable.

Since $\min\{16,12,10\}=10$, then a_3 leave the basic variable.

	x_1	x_2	s_1	e_2	a_2	a_3	RHS	Ratio
$-w$	2	0	0	1	0	4	-6	
s_1	$\frac{1}{4}$	0	1	0	0	$-\frac{1}{4}$	$\frac{3}{2}$	
a_2	-2	0	0	-1	1	-3	6	
x_2	1	1	0	0	0	1	10	

Since $w \neq 0$, then the origin LP must have no feasible solution.

Case II:

$$\begin{aligned} \min\quad & z = 2x_1 + 3x_2 \\ \text{s.t.}\quad & \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 \le 4 \\ & x_1 + 3x_2 \ge 20 \\ & x_1 + x_2 = 10 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Phase I problem:

$$\begin{aligned} \min\quad & w = a_2 + a_3 \\ \text{s.t.}\quad & \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 + s_1 = 4 \\ & x_1 + 3x_2 - e_2 + a_2 = 20 \\ & x_1 + x_2 + a_3 = 10 \\ & x_1, x_2, s_1, e_2, a_2, a_3 \ge 0 \end{aligned}$$

	x_1	x_2	s_1	e_2	a_2	a_3	RHS	Ratio
$-w$	0	0	0	0	1	1	0	
s_1	$\frac{1}{2}$	$\frac{1}{4}$	1	0	0	0	4	
a_2	1	3	0	-1	1	0	20	
a_3	1	1	0	0	0	1	10	

The initial table is

	x_1	x_2	s_1	e_2	a_2	a_3	RHS	Ratio
$-w$	-2	-4	0	1	0	0	-30	
s_1	$\frac{1}{2}$	$\frac{1}{4}$	1	0	0	0	4	16
a_2	1	3	0	-1	1	0	20	20/3
a_3	1	1	0	0	0	1	10	10

Since $\min\{-2,-4\}=-4$, then x_2 enter the basic variable.

Since $\min\{16,\frac{20}{3},10\}=\frac{20}{3}$ a_2 leave the basic variable

	x_1	x_2	s_1	e_2	a_2	a_3	RHS	Ratio
$-w$	$-\frac{2}{3}$	0	0	$-\frac{1}{3}$	$\frac{4}{3}$	0	$\frac{10}{3}$	
s_1	$\frac{5}{12}$	0	1	$\frac{1}{12}$	$-\frac{1}{12}$	0	$\frac{7}{3}$	28/5
x_2	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{20}{3}$	20
a_3	$\frac{2}{3}$	0	0	$\frac{1}{3}$	$-\frac{1}{3}$	1	$\frac{10}{3}$	5

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Since $\min\{-\frac{2}{3},-\frac{1}{3}\}=-\frac{2}{3}$, then x_1 enter the basic variable.

Since $\min\{\frac{28}{5},20,5\}=5$, then a_3 leave the basic variable.

	x_1	x_2	s_1	e_2	a_2	a_3	RHS	Ratio
$-w$	0	0	0	1	1	0	0	
s_1	0	0	1	$-\frac{1}{8}$	$\frac{1}{8}$	$-\frac{5}{8}$	$\frac{1}{4}$	
x_2	0	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	5	
x_1	1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	5	

Phase II:

$$\min \quad z=2x_1+3x_2$$
$$\text{s.t.} \quad s_1-\tfrac{1}{8}e_2=\tfrac{1}{4}$$
$$x_2-\tfrac{1}{2}e_2=5$$
$$x_1+\tfrac{1}{2}e_2=5$$
$$x_1,x_2,s_1,e_2\ge 0$$

	x_1	x_2	s_1	e_2	RHS	Ratio
$-z$	2	3	0	0	0	
s_1	0	0	1	$-\frac{1}{8}$	$\frac{1}{4}$	
x_2	0	1	0	$-\frac{1}{2}$	5	
x_1	1	0	0	$\frac{1}{2}$	5	

The initial table is

	x_1	x_2	s_1	e_2	RHS	Ratio
$-z$	0	0	0	$\frac{1}{2}$	-25	
s_1	0	0	1	$-\frac{1}{8}$	$\frac{1}{4}$	
x_2	0	1	0	$-\frac{1}{2}$	5	
x_1	1	0	0	$\frac{1}{2}$	5	

Hence, $x_1=x_2=5, z=25$

Case III:

$$\max \quad z=40x_1+10x_2+7x_5+14x_6$$
$$\text{s.t.} \quad x_1-x_2+2x_5=0$$
$$-2x_1+x_2-2x_5=0$$
$$x_1+x_3+x_5-x_6=3$$
$$2x_2+x_3+x_4+2x_5+x_6=4$$
$$x_i\ge 0 (i=1,2,3,4,5,6)$$

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Convert the standard form:

$$\max \quad z = 40x_1 + 10x_2 + 7x_5 + 14x_6$$

$$\text{s.t.} \quad x_1 - x_2 + 2x_5 + a_1 = 0$$

$$-2x_1 + x_2 - 2x_5 + a_2 = 0$$

$$x_1 + x_3 + x_5 - x_6 + a_3 = 3$$

$$2x_2 + x_3 + x_4 + 2x_5 + x_6 = 4$$

$$x_i \ge 0(i = 1,2,3,4,5,6), a_i \ge 0(i = 1,2,3)$$

Phase I:

$$\min \quad w = a_1 + a_2 + a_3$$

$$\text{s.t.} \quad x_1 - x_2 + 2x_5 + a_1 = 0$$

$$-2x_1 + x_2 - 2x_5 + a_2 = 0$$

$$x_1 + x_3 + x_5 - x_6 + a_3 = 3$$

$$2x_2 + x_3 + x_4 + 2x_5 + x_6 = 4$$

$$x_i \ge 0(i = 1,2,3,4,5,6), a_i \ge 0(i = 1,2,3)$$

Table Form

	x_1	x_2	x_3	x_4	x_5	x_6	a_1	a_2	a_3	RHS	Ratio
$-w$	0	0	0	0	0	0	1	1	1	0	
a_1	1	-1	0	0	2	0	1	0	0	0	
a_2	-2	1	0	0	-2	0	0	1	0	0	
a_3	1	0	1	0	1	-1	0	0	1	3	
x_4	0	2	1	1	2	1	0	0	0	4	

The initial table is

	x_1	x_2	x_3	x_4	x_5	x_6	a_1	a_2	a_3	RHS	Ratio
$-w$	0	0	-1	0	-1	1	0	0	0	-3	
a_1	1	-1	0	0	2	0	1	0	0	0	*
a_2	-2	1	0	0	-2	0	0	1	0	0	*
a_3	1	0	1	0	1	-1	0	0	1	3	3
x_4	0	2	1	1	2	1	0	0	0	4	4

Since $\min\{-1,-1\} = -1$. Then x_3 enter the basic variable.

Since $\min\{*,*,3,4\} = 3$, then a_3 leave the basic variable.

	x_1	x_2	x_3	x_4	x_5	x_6	a_1	a_2	a_3	RHS	Ratio
$-w$	1	0	0	0	0	0	0	0	1	0	
a_1	1	-1	0	0	2	0	1	0	0	0	
a_2	-2	1	0	0	-2	0	0	1	0	0	
x_3	1	0	1	0	1	-1	0	0	1	3	
x_4	-1	2	0	1	1	2	0	0	-1	1	

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Phase II:

$$\max \quad z = 10x_2 + 7x_5 + 14x_6$$

$$\text{s.t.} \quad -x_2 + 2x_5 + a_1 = 0$$

$$x_2 - 2x_5 + a_2 = 0$$

$$x_3 + x_5 - x_6 = 3$$

$$2x_2 + x_4 + x_5 + 2x_6 = 1$$

$$x_i \geq 0 (i = 2,3,4,5,6), a_i \geq 0 (i = 1,2)$$

Table Form

	x_2	x_3	x_4	x_5	x_6	a_1	a_2	RHS	Ratio
z	-10	0	0	-7	-14	0	0	0	
a_1	-1	0	0	2	0	1	0	0	*
a_2	1	0	0	-2	0	0	1	0	*
x_3	0	1	0	1	-1	0	0	3	*
x_4	2	0	1	1	2	0	0	1	0.5

Since $\min\{-10,-7,-14\} = -14$, then x_6 enter the basic variable.

Since $\min\{*,*,*,0.5\} = 0.5$, then x_4 leave the basic variable.

	x_2	x_3	x_4	x_5	x_6	a_1	a_2	RHS	Ratio
z	4	0	7	0	0	0	0	7	
a_1	0	0	0	2	0	1	0	0	
a_2	1	0	0	0	0	0	1	0	
x_3	1	1	0.5	1.5	0	0	0	3.5	
x_6	1	0	0.5	0.5	1	0	0	0.5	

Hence, $x_3 = 3.5, x_6 = 0.5, z = 7$.

Unrestricted-in-Sign Variables

$$\max \quad z = 30x_1 - 4x_2$$

$$\text{s.t.} \quad 5x_1 - x_2 \leq 30$$

$$x_1 \leq 5$$

$$x_1 \geq 0, x_2 \text{ urs}$$

Convert to

$$\max \quad z = 30x_1 - 4x_2' + 4x_2''$$

$$\text{s.t.} \quad 5x_1 - x_2' + x_2'' \leq 30$$

$$x_1 \leq 5$$

$$x_1, x_2', x_2'' \geq 0$$

Convert to the standard form:

$$\max \quad z - 30x_1 + 4x_2' - 4x_2'' = 0$$

$$\text{s.t.} \quad 5x_1 - x_2' + x_2'' + s_1 = 30$$

$$x_1 + s_2 = 5$$

$$x_1, x_2', x_2'', s_1, s_2 \geq 0$$

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Table Form

	x_1	x_2'	x_2''	s_1	s_2	RHS	Ratio
z	-30	4	-4	0	0	0	
s_1	5	-1	1	1	0	30	6
s_2	1	0	0	0	1	5	5

Since $\min\{-30,-4\}=-30$, then x_1 enter the basic variable.

Since $\min\{6,5\}=5$, then s_2 leave the basic variable.

	x_1	x_2'	x_2''	s_1	s_2	RHS	Ratio
z	0	4	-4	0	30	150	
s_1	0	-1	1	1	-5	5	5
x_1	1	0	0	0	1	5	*

Since $\min\{-4\}=-4$, then x_2'' enter the basic variable.

Since $\min\{5,*\}=5$, then s_1 leave the basic variable.

	x_1	x_2'	x_2''	s_1	s_2	RHS	Ratio
z	0	0	0	4	10	170	
x_2''	0	-1	1	1	-5	5	
x_1	1	0	0	0	1	5	

Hence, $x_1=5, x_2=x_2'-x_2''=0-5=-5, z=170$.

The Revised Simplex Method(修正單形法)

Assume that we are solving a max problem that has been prepared for solution by the Big-M method and that at this point, the LP has m constraints and n variables. Although some of these variables may be slack, surplus, or artificial, we choose to label them $x_1, x_2, \cdots, x_n$. Then the LP may be written

$$\begin{aligned} \max \quad & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ & \quad \vdots \qquad \vdots \qquad\qquad \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\ & x_i \ge 0 (i = 1, 2, \cdots, n) \end{aligned}$$

Matrix Form:

$$\begin{aligned} \max \quad & z = C^T X \\ \text{s.t.} \quad & AX = b \\ & X \ge 0 \end{aligned}$$

We can be written as

$$\begin{aligned} \max \quad & z = C_B^T X_B + C_N^T X_N \\ \text{s.t.} \quad & BX_B + NX_N = b \\ & X_B, X_N \ge 0 \end{aligned}$$

where

C_B^T is the $1\times m$ row vector whose elements are the coefficients of the basic variables.

C_N^T is the $1\times(n-m)$ row vector whose elements are the coefficients of the nonbasic variables.

The $m\times m$ matrix B is the matrix whose jth column is the column for basic variables.

N is the $m\times(n-m)$ matrix whose columns are the columns for the nonbasic variables.

X_B is the $m\times 1$ vector listing the basic variables.

X_N is the $(n-m)\times 1$ vector listing the nonbasic variables.

Example: Consider the max LP

$$\begin{aligned}
\max\quad & z = 60x_1 + 30x_2 + 20x_3 + 0s_1 + 0s_2 + 0s_3 \\
\text{s.t.}\quad & 8x_1 + 6x_2 + x_3 + s_1 = 48 \\
& 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20 \\
& 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8 \\
& x_1, x_2, x_3, s_1, s_2, s_3 \ge 0
\end{aligned}$$

Suppose that $X_B = \begin{bmatrix} s_1 \\ x_3 \\ x_1 \end{bmatrix}$, then

$$\max \qquad z = C_B^T X_B + C_N^T X_N = \begin{bmatrix} 0 & 20 & 60 \end{bmatrix}\begin{bmatrix} s_1 \\ x_3 \\ x_1 \end{bmatrix} + \begin{bmatrix} 30 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_2 \\ s_2 \\ s_3 \end{bmatrix}$$

$$\text{s.t.} \qquad BX_B + NX_N = b \equiv \begin{bmatrix} 1 & 1 & 8 \\ 0 & 1.5 & 4 \\ 0 & 0.5 & 2 \end{bmatrix}\begin{bmatrix} s_1 \\ x_3 \\ x_1 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 2 & 1 & 0 \\ 1.5 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_2 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 48 \\ 20 \\ 8 \end{bmatrix}$$

$$X_B, X_N \ge 0 \equiv x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$$

Assume that the LP problem is

$$\begin{aligned}
\max\quad & z = C_B^T X_B + C_N^T X_N \\
\text{s.t.}\quad & BX_B + NX_N = b \\
& X_B, X_N \ge 0
\end{aligned}$$

Then, the current tableau is

	X_B	X_N	RHS
z	$-C_B^T$	$-C_N^T$	0
X_B	B	N	b

$\Rightarrow$

	X_B	X_N	RHS
z	0	$C_B^T B^{-1} N - C_N^T$	$C_B^T B^{-1} b$
X_B	I	$B^{-1} N$	$B^{-1} b$

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Algorithm of the revised simplex method:

1. Compute $\bar{C}_N = C_B^T B^{-1} N - C_N^T = \left[\bar{c}_j\right]$. If $\bar{c}_j \geq 0$, $\forall j$, then the LP problem is optimal, stop; otherwise, choose q such that $\bar{c}_q = \min\{\bar{c}_j \mid \bar{c}_j < 0\}$, then x_q should enter the basic variable.
2. Compute $\bar{a}_q = B^{-1} a_q$ and $\bar{b} = B^{-1} b$. Choose p such that $\dfrac{\bar{b}_p}{\bar{a}_{pq}} = \min\left\{\dfrac{\bar{b}_i}{\bar{a}_{iq}} \mid \bar{a}_{iq} > 0\right\}$, then x_p should leave the basic variable.
3. Update X_B, X_N, B, N, go to step 1.

Example: Consider the LP

$$\begin{aligned} \max \quad & z = 3x_1 + 5x_2 \\ \text{s.t.} \quad & x_1 \leq 4 \\ & 2x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 18 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Then, the standard form is

$$\begin{aligned} \max \quad & z = 3x_1 + 5x_2 \\ \text{s.t.} \quad & x_1 + s_1 = 4 \\ & 2x_2 + s_2 = 12 \\ & 3x_1 + 2x_2 + s_3 = 18 \\ & x_1, x_2, s_1, s_2, s_3 \geq 0 \end{aligned}$$

Iteration 0

Since $X_B = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$ and $X_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = B^{-1}$.

Since $\bar{C}_N^T = C_B^T B^{-1} N - C_N^T = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 5 \end{bmatrix} = \begin{bmatrix} -3 & -5 \end{bmatrix}$, then x_2 enter the basic variable.

Since $\bar{a}_2 = B^{-1} a_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$, $\bar{b} = B^{-1} b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$, and

$\min\left\{\times, \dfrac{12}{2}, \dfrac{18}{2}\right\} = 6$, then s_2 leave the basic variable.

Iteration 1

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Since $X_B = \begin{bmatrix} s_1 \\ x_2 \\ s_3 \end{bmatrix}$ and $X_N = \begin{bmatrix} x_1 \\ s_2 \end{bmatrix}$, then $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}$.

Since $\bar{C}_N^T = C_B^T B^{-1} N - C_N^T = \begin{bmatrix} 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 0 \end{bmatrix} = \begin{bmatrix} -3 & \frac{5}{2} \end{bmatrix}$, then x_1 enter the basic variable.

Since $\bar{a}_1 = B^{-1} a_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$, $\bar{b} = B^{-1} b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix}$, and

$\min\left\{ \times, \times, \frac{6}{2} \right\} = 3$, then s_3 leave the basic variable.

Iteration 2

Since $X_B = \begin{bmatrix} s_1 \\ x_2 \\ x_1 \end{bmatrix}$ and $X_N = \begin{bmatrix} s_2 \\ s_3 \end{bmatrix}$, then $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 3 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$.

Since $\bar{C}_N^T = C_B^T B^{-1} N - C_N^T = \begin{bmatrix} 0 & 5 & 3 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 1 \end{bmatrix}$, then

$$X_B = \begin{bmatrix} s_1 \\ x_2 \\ x_1 \end{bmatrix} = B^{-1} b = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}, \text{and } z_{\text{optimal}} = C_B^T B^{-1} b = C_B^T X_B = \begin{bmatrix} 0 & 5 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} = 36.$$

Product Form of the Inverse

Consider a basis (matrix) B composed of the columns $a_{B_1}, a_{B_2}, \cdots, a_{B_m}$ and suppose that B^{-1} is known.

$$B = \begin{bmatrix} a_{B_1} & a_{B_2} & \cdots & a_{B_m} \end{bmatrix}$$

Suppose that the nonbasic column a_k replaces a_{B_r}, resulting in the new basis (matrix) B_{new}.

$$B_{\text{new}} = \begin{bmatrix} a_{B_1} & a_{B_2} & \cdots & a_{B_{r-1}} & a_k & a_{B_{r+1}} & \cdots & a_{B_m} \end{bmatrix}$$

Noting that $a_k = B\bar{a}_k$ since $\bar{a}_k = B^{-1}a_k$ and $a_{B_i} = Be_i$ where e_i is a vector of zeros except for 1 at the ith position, we have

$$\begin{aligned} B_{\text{new}} &= \begin{bmatrix} a_{B_1} & a_{B_2} & \cdots & a_{B_{r-1}} & a_k & a_{B_{r+1}} & \cdots & a_{B_m} \end{bmatrix} \\ &= \begin{bmatrix} Be_1 & Be_2 & \cdots & Be_{r-1} & B\bar{a}_k & Be_{r+1} & \cdots & Be_m \end{bmatrix} \\ &= B\begin{bmatrix} e_1 & e_2 & \cdots & e_{r-1} & \bar{a}_k & e_{r+1} & \cdots & e_m \end{bmatrix} \\ &= BT \end{aligned}$$

where T is the identity with the rth column replaced by $\bar{a}_k$, i.e.,

rth column

$$T = \begin{bmatrix} 1 & 0 & \cdots & 0 & \bar{a}_{1k} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \bar{a}_{2k} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \bar{a}_{rk} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \bar{a}_{mk} & 0 & \cdots & 1 \end{bmatrix} \leftarrow r\text{th row}$$

Since

$$\left[\begin{array}{cccccccc|cccccccc} 1 & 0 & \cdots & 0 & \bar{a}_{1k} & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \bar{a}_{2k} & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \bar{a}_{rk} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \bar{a}_{mk} & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccccccc|cccccccc} 1 & 0 & \cdots & 0 & \bar{a}_{1k} & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \bar{a}_{2k} & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \dfrac{1}{\bar{a}_{rk}} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \bar{a}_{mk} & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{array}\right]$$

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$$\sim\left[\begin{array}{cccccccc|cccccccc} & & & & & & & & 1 & 0 & \cdots & 0 & -\frac{\bar{a}_{1k}}{\bar{a}_{rk}} & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & -\frac{\bar{a}_{2k}}{\bar{a}_{rk}} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & 0 & 0 & \cdots & 0 & \frac{1}{\bar{a}_{rk}} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & 0 & 0 & \cdots & 0 & -\frac{\bar{a}_{mk}}{\bar{a}_{rk}} & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & & & & & & & & \end{array}\right]$$

,then the inverse of T is

$$T^{-1}=\begin{bmatrix} 1 & 0 & \cdots & 0 & -\frac{\bar{a}_{1k}}{\bar{a}_{rk}} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & -\frac{\bar{a}_{2k}}{\bar{a}_{rk}} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \frac{1}{\bar{a}_{rk}} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & -\frac{\bar{a}_{mk}}{\bar{a}_{rk}} & 0 & \cdots & 1 \end{bmatrix}$$

Therefore, $B_{\text{new}}^{-1}=(BT)^{-1}=T^{-1}B^{-1}=EB^{-1}$.

Example: Consider the LP

$$\begin{array}{lllll} \max & z=3x_1+5x_2 & & & \\ \text{s.t.} & x_1 & +s_1 & & =4 \\ & 2x_2 & & +s_2 & =12 \\ & 3x_1+2x_2 & & +s_3 & =18 \\ & x_1,x_2,s_1,s_2,s_3\ge 0 & & & \end{array}$$

1. If $X_B=\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$, then $B_1=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=B_1^{-1}$.

2. If $X_B=\begin{bmatrix} s_1 \\ x_2 \\ s_3 \end{bmatrix}$, i.e., x_2 replace s_2, then

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$$\bar{a}_2 = B_1^{-1}a_2 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$B_2^{-1} = E_1 B_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

3. If $X_B = \begin{bmatrix} s_1 \\ x_2 \\ x_1 \end{bmatrix}$, i.e., x_1 replace s_3, then

$$\bar{a}_1 = B_2^{-1}a_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$B_3^{-1} = E_2 B_2^{-1} = \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Example: The LP is

max $\quad z = 3x_1 + x_2 + x_3$

s.t. $\quad x_1 + x_2 + x_3 \le 6$

$2x_1 - x_3 \le 4$

$x_2 + x_3 \le 2$

$x_1, x_2, x_3 \ge 0$

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Then, the standard form is

$$\begin{aligned} \max \quad & z = 3x_1 + x_2 + x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 + s_1 = 6 \\ & 2x_1 - x_3 + s_2 = 4 \\ & x_2 + x_3 + s_3 = 2 \\ & x_1, x_2, x_3, s_1, s_2, s_3 \geq 0 \end{aligned}$$

Iteration 0

Since $X_B = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$ and $X_N = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, then $B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = B_1^{-1}$.

Since $\overline{C}_N^T = C_B^T B^{-1} N - C_N^T = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -1 & -1 \end{bmatrix}$, then x_1 enter the basic variable.

Since $\overline{a}_1 = B_1^{-1} a_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\overline{b} = B^{-1} b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$, and $\min \left\{ \frac{6}{1}, \frac{4}{2}, \times \right\} = 2$, then s_2 leave the basic variable.

Iteration 1

Since $X_B = \begin{bmatrix} s_1 \\ x_1 \\ s_3 \end{bmatrix}$ and $X_N = \begin{bmatrix} x_2 \\ x_3 \\ s_2 \end{bmatrix}$, then $E = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B_2^{-1} = E B_1^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Since $\overline{C}_N^T = C_B^T B^{-1} N - C_N^T = \begin{bmatrix} 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -\frac{5}{2} & \frac{3}{2} \end{bmatrix}$, then x_3 enter the basic variable.

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Since $\bar{a}_3 = B_2^{-1}a_3 = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$, $\bar{b} = B_2^{-1}b = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$, and

$\min\left\{ \frac{4}{\frac{3}{2}}, \times, \frac{2}{1} \right\} = 2$, then s_3 leave the basic variable.

Iteration 2

Since $X_B = \begin{bmatrix} s_1 \\ x_1 \\ x_3 \end{bmatrix}$ and $X_N = \begin{bmatrix} x_2 \\ s_2 \\ s_3 \end{bmatrix}$, then $E = \begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$ and

$$B_3^{-1} = EB_2^{-1} = \begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $\bar{C}_N^T = C_B^T B^{-1} N - C_N^T = \begin{bmatrix} 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{3}{2} & \frac{5}{2} \end{bmatrix}$, then

$$X_B = \begin{bmatrix} s_1 \\ x_1 \\ x_3 \end{bmatrix} = B_3^{-1}b = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \text{and } z_{\text{optimal}} = C_B^T B^{-1} b = C_B^T X_B = \begin{bmatrix} 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = 11.$$

The LINDO Computer Package

1. Setup the LINDO package
2. Execute Lindow32

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Appendix 1 (Proof of Fundamental Theorem of Linear Programming)

Proof of (1):

Denote the columns of A by $a_1, a_2, \cdots, a_n$. Suppose $X = [x_1 \quad x_2 \quad \cdots \quad x_n]^T$ is a feasible solution. Then, in terms of the columns of A, this solution satisfies:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Assume that exactly p of the variables x_i are greater than zero, and for convenience, that they are first p variables. Thus,

$$a_1x_1 + a_2x_2 + \cdots + a_px_p = b$$

There are two cases, corresponding as to whether the set $a_1, a_2, \cdots, a_p$ is linearly independent or linearly dependent.

CASE 1: Assume $a_1, a_2, \cdots, a_p$ is linearly independent. Then $p \le m$.

If $p = m$, the solution is basic and the proof is complete.

If $p < m$, then, since A has rank m, $m - p$ vectors can be found from the remaining $n - p$ vectors so that the resulting set of m vectors is linearly independent. Assigning the value zero to the corresponding $m - p$ variables yields a (degenerate) basic feasible solution.

CASE 2: Assume $a_1, a_2, \cdots, a_p$ is linearly dependent. Then there is a nontrivial linear combination of these vectors that is zero. Thus, there are constants $y_1, y_2, \cdots, y_p$, at least one of which can be assumed to be positive, such that

$$a_1y_1 + a_2y_2 + \cdots + a_py_p = 0$$

Multiplying this equation by a scalar ε and subtracting it from $a_1x_1 + a_2x_2 + \cdots + a_px_p = b$, we obtain

$$a_1(x_1 - \varepsilon y_1) + a_2(x_2 - \varepsilon y_2) + \cdots + a_p(x_p - \varepsilon y_p) = b$$

This equation holds for every ε, and for each ε the components $x_i - \varepsilon y_i$ correspond to a solution of the linear equalities—although they may violate $x_i - \varepsilon y_i \ge 0$. Denoting $y = [y_1 \quad y_2 \quad \cdots \quad y_p \quad 0 \quad 0 \quad \cdots \quad 0]^T$, we see that for any ε

$$X - \varepsilon Y$$

is a solution to the equalities. For $\varepsilon = 0$, this reduces to the origin feasible solution. As ε is increased from zero, the various components increase, decrease, or remain constant, depending upon whether the corresponding y_i is negative, positive, or zero. Since we

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assume at least one y_i is positive, at least one component will decrease as ε is increased. Increase ε to the first point where one or more components become zero. Specifically, set

$$\varepsilon = \min\left\{\frac{x_i}{y_i} : y_i > 0\right\}$$

For this value of ε the solution given by $x - \varepsilon y$ is feasible and has at most $p-1$ positive variables. Repeating this process if necessary, we can eliminate positive variables until we have a feasible solution with corresponding columns that are linearly independent. AT that point CASE 1 applies.

Proof of (2):

Let $X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$ be an optimal feasible solution and, as in the proof of (1) above, suppose there are exactly p positive variables $a_1, a_2, \cdots, a_p$. Again there are two cases; and CASE 1, corresponding to linear independence, is exactly the same as before.

CASE2 also goes exactly the same before, but it must be shown that for any ε the solution $X - \varepsilon Y$ is optimal. To show this, note that the value of the solution $X - \varepsilon Y$ is

$$C^T X - \varepsilon C^T Y$$

For ε sufficiently small in magnitude, $X - \varepsilon Y$ is a feasible solution for positive or negative values of ε. Thus, we conclude that $C^T Y = 0$. For, if $C^T y \neq 0$, an ε of small magnitude and proper sign could be determined so as to render $C^T X - \varepsilon C^T Y$ smaller than $C^T X$ while maintaining feasibility. This would violate the assumption of optimality of X and hence we must have $C^T Y = 0$.

Having established that the new feasible solution with fewer positive components is also optimal, the remainder of the proof may be completed exactly as in part (1).

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Appendix 2 (Proof of Equivalence of Extreme Points and Basic Feasible Solution)

Suppose first that $X=\begin{bmatrix} x_1 & x_2 & \cdots & x_m & 0 & 0 & \cdots & 0 \end{bmatrix}^T$ is a basic feasible solution to $AX=b$ and $X\ge 0$.. Then

$$a_1x_1+a_2x_2+\cdots+a_mx_m=b$$

where $a_1,a_2,\cdots,a_m$, the first m columns of A, are linearly independent. Suppose that X could be expressed as a convex combination of two other points in K; say, $X=\alpha Y+(1-\alpha)Z$, $0<\alpha<1$, $Y\ne Z$. Since all components of X,Y,Z are nonnegative and since $0<\alpha<1$, it follows immediately that the last $n-m$ components of Y and Z are zero. Thus, in particular, we have

$$a_1y_1+a_2y_2+\cdots+a_my_m=b$$

and

$$a_1z_1+a_2z_2+\cdots+a_mz_m=b$$

Since the vectors $a_1,a_2,\cdots,a_m$ are linearly independent, it follows that $X=Y=Z$ and hence X is an extreme point of K.

Conversely, assume that X is an extreme point of K. Let us assume that the nonzero components of X are the first k components. Then

$a_1x_1+a_2x_2+\cdots+a_mx_m=b$

with $x_i>0$, $i=1,2,\cdots,k$. To show that X is a basic feasible solution it must be shown that the vectors $a_1,a_2,\cdots,a_m$ are linearly independent. We do this by contradiction. Suppose that $a_1,a_2,\cdots,a_m$ are linearly dependent. Then there is a nontrivial linear combination that is zero:

$$a_1y_1+a_2y_2+\cdots+a_ky_k=0$$

Define the n-vector $Y=\begin{bmatrix} y_1 & y_2 & \cdots & y_k & 0 & 0 & \cdots & 0 \end{bmatrix}$. Since $x_i>0$, , it is possible to select ε such that

$$X+\varepsilon Y\ge 0 \quad \text{and} \quad X-\varepsilon Y\ge 0$$

We then have $X=\frac{1}{2}\left(X+\varepsilon Y\right)+\frac{1}{2}\left(X-\varepsilon Y\right)$ which expresses X as a convex combination of two distinct vectors in K. This cannot occur, since X is an extreme point of K. Thus, $a_1,a_2,\cdots,a_m$ are linearly independent and X is a basic feasible solution.