Matrix Algebra(矩陣代數)

Definition of Matrix

A **matrix(矩陣)** is any rectangular array of numbers with m rows(列) and n columns(行).

Example:

$$A=\begin{bmatrix}3&2&4\\6&0&1\end{bmatrix},\quad B=\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix},\quad C=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad D=\begin{bmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{bmatrix}$$

are all matrices.

Example: The structural steel plant is expected to earn \$4.0 million if the rival firm does not enter the market and \$2.0 million if it does. The stainless steel plant is expected to earn \$3.1 million whether or not the competitor enters the market, but the diversified plant will earn \$3.5 million or \$2.6 million according to whether or not a competitive plant is erected.

$$A=\begin{bmatrix}4.0&3.1&3.5\\2.0&3.1&2.6\end{bmatrix}$$

Notation:

If a matrix A has m rows and n columns, we call A an $m\times n$ matrix. We refer to $m\times n$ as the **order(階)** of the matrix. A typical $m\times n$ matrix A nay be written as

$$A_{m\times n}=[a_{ij}]_{m\times n}=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&\cdots&\vdots\\a_{m1}&a_{m2}&\cdots&a_{mn}\end{bmatrix}$$

Remark: If $m=n$, then the matrix is a **square matrix(方陣)**.

Example: $A=\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}$ is an 2×3 matrix.

The number in the i th row and j th column of A is called the ij th **element(元素)(or entry)** of A and is written a_{ij}.

For example, if

$$A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$$

then $a_{11}=1$, $a_{23}=6$, and $a_{31}=7$.

Definition of Equality

Two matrices $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$ are **equal(相等)** if and only if A and B are of the same order and for all i and j, $a_{ij}=b_{ij}$.

For example, if

$$A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} \text{ and } B=\begin{bmatrix}x & y\\ w & z\end{bmatrix}$$

then $A=B$ if and only if $x=1$, $y=2$, $w=3$, and $z=4$.

Remarks:

1. Any matrix with one column may be thought of as a **column vector(行向量)**. The number of rows in a column vector is the dimension of the column vector. Thus,
$$\begin{bmatrix}1\\ 2\end{bmatrix}$$
may be thought of as a 2×1 matrix or a two-dimension column vector. $\mathbb{R}^m$ will denote the set of all m-dimensional column vectors.
2. In analogous fashion, we can think of any matrix with only one row as a **row vector(列向量)**.
3. An m-dimensional vector in which all elements equal zero is called a **zero vector(零向量)**.
4. Any m-dimensional vector corresponds to a **directed line segment(方向直線線段)** in the m-dimensional plane.

The Scalar Product(純量積) of Two Vectors

Suppose we have a row vector $\vec{u}=\begin{bmatrix}u_1 & u_2 & \cdots & u_n\end{bmatrix}$ and a column vector

$$\vec{v}=\begin{bmatrix}v_1\\ v_2\\ \vdots\\ v_n\end{bmatrix}$$

of the same dimension. The **scalar product (inner product)(內積)** of $\vec{u}$ and $\vec{v}$ (written $\vec{u}\cdot\vec{v}$) is the number $u_1v_1+u_2v_2+\cdots+u_nv_n$.

For example, if

$$\vec{u}=\begin{bmatrix}1 & 2 & 3\end{bmatrix} \text{ and } \vec{v}=\begin{bmatrix}2\\ 1\\ 2\end{bmatrix}$$

then $\mathrm{u}\cdot\mathrm{v}=1(2)+2(1)+3(2)=10$.

Remark 1: Two zero vectors are **perpendicular**(垂直) if and only if their scalar product equals 0.

Remark 2: $\vec{u}\cdot\vec{v}=\|\vec{u}\|\|\vec{v}\|\cos\theta$, where $\|\vec{u}\|$ is the length of the vector $\vec{u}$ and θ is the angle between the vectors $\vec{u}$ and $\vec{v}$.

Matrix Operations(矩陣運算)

1. **The Scalar Multiple of a Matrix(純量乘矩陣)**

   Given any matrix A and any number c, the matrix cA is obtained from the matrix A by multiplying each element of A by c.

   Example: If $A=\begin{bmatrix}1 & 2\\ -1 & 0\end{bmatrix}$, then $3A=\begin{bmatrix}3 & 6\\ -3 & 0\end{bmatrix}$

2. **Addition of Two Matrices(矩陣相加)**

   Let $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$ be two matrices with the **same order**. Then the matrix $C=A+B$ is defined to be the $m\times n$ matrix whose ij th element is $a_{ij}+b_{ij}$.

   Example: If $A=\begin{bmatrix}1 & 2 & 3\\ 0 & -1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}-1 & -2 & -3\\ 2 & 1 & -1\end{bmatrix}$, then $A+B=\begin{bmatrix}0 & 0 & 0\\ 2 & 0 & 0\end{bmatrix}$

3. **The Transpose of a Matrix(轉置矩陣)**

   Given any $m\times n$ matrix

$$A=\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}$$

   the transport of A (written A^T) is the $n\times m$ matrix

$$A^T=\begin{bmatrix}a_{11} & a_{21} & \cdots & a_{m1}\\ a_{12} & a_{22} & \cdots & a_{m2}\\ \vdots & \vdots & & \vdots\\ a_{1n} & a_{2n} & \cdots & a_{mn}\end{bmatrix}$$

   Example: If $A=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}$, then $A^T=\begin{bmatrix}1 & 4\\ 2 & 5\\ 3 & 6\end{bmatrix}$

   Remark: $(A^T)^T=A$

4. **Matrix Multiplication(矩陣乘法)**

   The matrix product $C=AB$ of A and B is the $m\times n$ matrix C whose ij th element is determined as follows:

   ij th element of $C=$ scalar product of row i of $A\times$column j of B

Example: If $A=\begin{bmatrix}1 & 1 & 2\\ 2 & 1 & 3\end{bmatrix}$ and $B=\begin{bmatrix}1 & 1\\ 2 & 3\\ 1 & 2\end{bmatrix}$, then $AB=\begin{bmatrix}5 & 8\\ 7 & 11\end{bmatrix}$

Example: If $A=\begin{bmatrix}3\\ 4\end{bmatrix}$ and $B=\begin{bmatrix}1 & 2\end{bmatrix}$, then $AB=\begin{bmatrix}3 & 6\\ 4 & 8\end{bmatrix}$, and $BA=11$

Remark: $AB \neq BA$. (not commutative)

Properties of Matrix Multiplication(矩陣乘法性質)

1. Row i of $AB=$ (row i of A) B

   If $A=\begin{bmatrix}1 & 1 & 2\\ 2 & 1 & 3\end{bmatrix}$ and $B=\begin{bmatrix}1 & 1\\ 2 & 3\\ 1 & 2\end{bmatrix}$

   Then row 2 of the 2×2 matrix AB is equal to

$$\begin{bmatrix}2 & 1 & 3\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & 3\\ 1 & 2\end{bmatrix}=\begin{bmatrix}7 & 11\end{bmatrix}$$

2. Column j of $AB=$ A (column j of B)

   The first column of AB is

$$\begin{bmatrix}1 & 1 & 2\\ 2 & 1 & 3\end{bmatrix}\begin{bmatrix}1\\ 2\\ 1\end{bmatrix}=\begin{bmatrix}5\\ 7\end{bmatrix}$$

3. Matrix multiplication is **associative**. That is, $A(BC)=(AB)C$.
4. Matrix multiplication is **distributive**. That is, $A(B+C)=AB+AC$.

Matrices and Systems of Linear Equations(聯立方程式)

Consider a system of linear equations given by

$$\begin{aligned}a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n&=b_1\\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n&=b_2\\ \vdots \qquad \vdots \qquad\qquad \vdots&\\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n&=b_m\end{aligned}$$

Let $A=\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}$, $x=\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}$, $b=\begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_m\end{bmatrix}$, then above system of linear equations may be written as $Ax=b$.

Example: Consider a system of linear equations given by

$$\begin{aligned}3x_1-5x_2+7x_3&=10\\-2x_1-4x_2+x_3&=0\\x_1+2x_2+3x_3&=4\\5x_1-x_2+2x_3&=-3\end{aligned}$$

Then the matrix form is

$$\begin{bmatrix}3&-5&7\\-2&-4&1\\1&2&3\\5&-1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}10\\0\\4\\-3\end{bmatrix}$$

Definition of Solution

A **solution(解)** to a linear system of m equations in n unknown is a set of values for the unknowns that satisfy each of the system's equations.

For example,

$\mathrm{x}=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$ is a solution to the linear system $\begin{cases}x_1+2x_2=5\\2x_1-x_2=0\end{cases}$.

The Gauss-Jordan Method for Solving Systems of Linear Equations

Using the Gauss-Jordan method, we show that any system of linear equations must satisfy one of the following three cases:

Case 1 The system has no solution.

Case 2 The system has a unique solution.

Case 3 The system has an infinite number of solutions.

Elementary Row Operations

An **elementary row operation (ERO)(基本列運算)** transforms a given matrix A into a new matrix A' via one of the following operations.

Type 1 ERO:

A' is obtained by multiplying any row of A by a nonzero scalar.

Example: $A=\begin{bmatrix}1&2&3&4\\1&3&5&6\\0&1&2&3\end{bmatrix}$, $A'=\begin{bmatrix}1&2&3&4\\3&9&15&18\\0&1&2&3\end{bmatrix}$

Type 2 ERO:

For some $j\neq i$, let row j of $A'=c(\text{row } i \text{ of } A)+(\text{row } j \text{ of } A)$

Example: $A=\begin{bmatrix}1&2&3&4\\1&3&5&6\\0&1&2&3\end{bmatrix}$, $A'=\begin{bmatrix}1&2&3&4\\1&3&5&6\\4&13&22&27\end{bmatrix}$

Type 3 ERO:

Interchange any two rows of A.

Example: $A=\begin{bmatrix}1&2&3&4\\1&3&5&6\\0&1&2&3\end{bmatrix}$, $A'=\begin{bmatrix}0&1&2&3\\1&3&5&6\\1&2&3&4\end{bmatrix}$

Finding a Solution by the Gauss-Jordan Method

1. To solve $Ax=b$, write down the augmented matrix $A\,|\,b$.
2. At any stage, define a current row, current column, and current entry. Begin with row 1 as the current row, column 1 as the current, and a_{11} as the current entry.
   (a) If a_{11} is nonzero, then use EROs to transform column 1 to

$$\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}$$

   Then obtain the new current row, column, and entry by moving down one row and one column to the right, and go to step 3.

   (b) If a_{11} equals 0, then do a Type 3 ERO involving the current row and any row that contains a nonzero number in the current column. Use EROs to transform column 1 to

$$\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}$$

   Then obtain the new row, column, and entry by moving down one row and one column to the right. Go to step 3.

   (c) If there are no nonzero numbers in the first column, then obtain a new current column and entry by moving one column to the right. Then go to step 3.
3. (a) If the new current entry is nonzero, then use EROs to transform it to 1 and the rest of the current column's entries to 0. When finished, obtain the new current row, column, and entry. If this is impossible, then stop. Otherwise, repeat step 3.
   (b) If the current entry is 0, then do a Type 3 ERO with the current row and any row that contains a nonzero number in the current column. Then use EROs to transform that current entry to 1 and the rest of the current column's entries to 0. When finished, obtain the new current row, column, and entry. If this is impossible, then stop. Otherwise, repeat step 3.
   (c) If the current column has no nonzero numbers below the current row, then obtain the new current column and entry, and repeat step 3. If it is impossible, then stop.
4. Write down the system of equations $A'x=b'$ that corresponds to the matrix $A'\,|\,b'$ obtained when step 3 is completed. Then $A'x=b'$ will have the same set of solutions as $Ax=b$.

Remark: $Ax=b \xrightarrow{\text{ERO}} A'x=b'$

Example: Find the solution to the following linear system:

$$2x_1 + 2x_2 + x_3 = 9$$
$$2x_1 - x_2 + 2x_3 = 6$$
$$x_1 - x_2 + 2x_3 = 5$$

The augmented matrix representation is

$$A|b = \left[\begin{array}{ccc|c} 2 & 2 & 1 & 9 \\ 2 & -1 & 2 & 6 \\ 1 & -1 & 2 & 5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 2 & 1 & 9 \\ 2 & -1 & 2 & 6 \\ 1 & -1 & 2 & 5 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1/2 & 9/2 \\ 2 & -1 & 2 & 6 \\ 1 & -1 & 2 & 5 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1/2 & 9/2 \\ 0 & -3 & 1 & -3 \\ 1 & -1 & 2 & 5 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1/2 & 9/2 \\ 0 & -3 & 1 & -3 \\ 0 & -2 & 3/2 & 1/2 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1/2 & 9/2 \\ 0 & 1 & -1/3 & 1 \\ 0 & -2 & 3/2 & 1/2 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 5/6 & 7/2 \\ 0 & 1 & -1/3 & 1 \\ 0 & -2 & 3/2 & 1/2 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 5/6 & 7/2 \\ 0 & 1 & -1/3 & 1 \\ 0 & 0 & 5/6 & 5/2 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 5/6 & 7/2 \\ 0 & 1 & -1/3 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -1/3 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Hence, the solution of the system of equations is

$$x_1 = 1$$
$$x_2 = 2$$
$$x_3 = 3$$

Exercise: Find the solution to the following linear system:

$$x_1 - x_2 + 5x_3 = -6$$
$$3x_1 + 3x_2 - x_3 = 10$$
$$x_1 + 3x_2 + 2x_3 = 5$$

The solution of the system of equations is

$$x_1 = 1$$
$$x_2 = 2$$
$$x_3 = -1$$

Example: Find all solutions to the following linear system:

$$x_1 + 2x_2 = 3$$
$$2x_1 + 4x_2 = 4$$

Solution: Since $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4 & 4 \end{array}\right] \xrightarrow{\text{ERO}} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 2 \end{array}\right]$

Hence, the linear system has no solution.

Example: Apply the Gauss-Jordan method to the following linear system:

$$x_1 + x_2 \quad\quad = 1$$
$$x_2 + x_3 = 3$$
$$x_1 + 2x_2 + x_3 = 4$$

Solution: Since $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 1 & 2 & 1 & 4 \end{array}\right] \xrightarrow{\text{ERO}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Hence, the linear system has an infinite number of solutions.

Basic Variables and Solutions to Linear Equation Systems

After the Gauss-Jordan method has been applied to any linear system, a variable that appears with a coefficient of 1 in a single equation and a coefficient of 0 in all other equations is called a **basic variable (BV)(基變數)**.

Any variable that is not a basic variable is called a **nonbasic variable (NBV)(非基變數)**.

Example: Consider

$$A' \mid b' = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Then $BV = \{x_1, x_2, x_3\}$ and $NBV = \{x_4, x_5\}$.

Linear Independence and Linear Dependence

A **linear combination(線性組合)** of the vectors in V is any vector of the form

$c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + \cdots + c_k\mathrm{v}_k$, where $c_1, c_2, \cdots, c_k$ are arbitrary scalars.

Example: If $V = \{[1 \quad 2], [2 \quad 1]\}$, then

$$\begin{aligned} 2v_1 - v_2 &= 2[1 \quad 2] - [2 \quad 1] = [0 \quad 3] \\ v_1 + 3v_2 &= [1 \quad 2] + 3[2 \quad 1] = [7 \quad 5] \\ 0v_1 + 3v_2 &= 0[1 \quad 2] + 3[2 \quad 1] = [6 \quad 3] \end{aligned}$$

are linear combinations of vectors in V.

Definition of Linearly Independent

A set V of m-dimensional vectors is **linearly independent(線性獨立)** if the only linear combination of vectors in V that equals $\vec{0}$ is the trivial linear combination.

Example: Since $c_1[1 \quad 0] + c_2[0 \quad 1] = [0 \quad 0] \Leftrightarrow c_1 = c_2 = 0$. Then the set of vectors $V = \{[1,0],[0,1]\}$ is a linearly independent set of vectors.

Definition of Linearly Dependent

A set V of m-dimensional vectors is **linearly dependent(線性相關)** if there is a nontrivial linear combination of vectors in V that adds up to $\vec{0}$.

Example: Since $c_1[1 \quad 2] + c_2[2 \quad 4] = [0 \quad 0] \Leftrightarrow c_1 = -2c_2$, then there is a nontrivial linear combination with $c_1 = 2$ and $c_2 = -1$. Hence, the set of vector $V = \{[1,2],[2,4]\}$ is a linearly dependent set of vectors.

Definition of the Rank of a Matrix

The **rank(秩)** of A is the number of vectors in the largest linearly independent subset of $\mathbb{R}$.

For examples:

1. The rank of $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is 0.
2. The rank of $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is 1.
3. The rank of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is 2.

Example Using Gauss-Jordan Method to Find Rank of Matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{Thus, rank } A = 3.$$

Definition of the Inverse of a Matrix

The **diagonal** elements of a **square matrix** are those elements a_{ij} such that $i = j$.

A **square matrix** for which all diagonal elements are equal to 1 and all nondiagonal elements are equal to 0 is called an **identity matrix(單位矩陣)**.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For a given $m \times m$ matrix A, the $m \times m$ matrix $B = A^{-1}$ is the **inverse(反矩陣)** of A if

$$BA = AB = I_m$$

Example: Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$. Since

$$\begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -5 & 1 & -7 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 1 \\ -5 & 1 & -7 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then $A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -5 & 1 & -7 \\ 1 & 0 & 2 \end{bmatrix}$

Use the Gauss-Jordan Method to Invert a Matrix

Example: $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

Solution:

$$\left[\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 5/2 & 1/2 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 5/2 & 1/2 & 0 \\ 0 & 1/2 & -1/2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 5/2 & 1/2 & 0 \\ 0 & 1 & -1 & 2 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

Hence, $A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

Example: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Since $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right]$

Hence matrix A does not have inverse.

Example: Find the inverse of $A=\begin{bmatrix}1 & 0 & 1\\ 2 & -2 & -1\\ 3 & 0 & 0\end{bmatrix}$.

Solution: $A^{-1}=\begin{bmatrix}0 & 0 & 1/3\\ -1/2 & -1/2 & 1/2\\ 1 & 0 & -1/3\end{bmatrix}$

Using Matrix Inverses to Solve Linear Systems

$$A\text{x}=b\Rightarrow \text{x}=A^{-1}b$$

For example, to solve $\begin{matrix}2x_1+5x_2=7\\ x_1+3x_2=4\end{matrix}$ write the matrix representation: $\begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}7\\ 4\end{bmatrix}$

$$A=\begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix}\Rightarrow A^{-1}=\begin{bmatrix}3 & -5\\ -1 & 2\end{bmatrix}$$

$$\text{x}=\begin{bmatrix}x_1\\ x_2\end{bmatrix}=A^{-1}b=\begin{bmatrix}3 & -5\\ -1 & 2\end{bmatrix}\begin{bmatrix}7\\ 4\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix}$$

Example: To solve $AX=b$, where

$$A=\begin{bmatrix}1 & 2 & 3\\ 3 & 1 & 2\\ 2 & 0 & 1\end{bmatrix}, b=\begin{bmatrix}18\\ 23\\ 13\end{bmatrix}$$

Solution: Since $A^{-1}=\begin{bmatrix}-1/3 & 2/3 & -1/3\\ -1/3 & 5/3 & -7/3\\ 2/3 & -4/3 & 5/3\end{bmatrix}$, then $X=A^{-1}b=\begin{bmatrix}-1/3 & 2/3 & -1/3\\ -1/3 & 5/3 & -7/3\\ 2/3 & -4/3 & 5/3\end{bmatrix}\begin{bmatrix}18\\ 23\\ 13\end{bmatrix}=\begin{bmatrix}5\\ 2\\ 3\end{bmatrix}$

Definition of Determinants

Associated with any square matrix A is a number called the **determinants(行列式)** of A (often abbreviated as $\det A$ or $|A|$).

1. For a 1×1 matrix $A=[a_{11}]$,

$$|A|=a_{11}$$

2. For a 2×2 matrix $A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$,

$$|A|=a_{11}a_{22}-a_{12}a_{21}$$

3. For a 3×3 matrix $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

$$|A| = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{23}a_{12} - a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{21}a_{12}$$

Definition of Minor

If A is an $m\times m$ matrix, then for any values of i and j, the ijth **minor(餘因子)** of A (written A_{ij}) is the $(m-1)\times(m-1)$ submatrix(子矩陣) of A obtained by deleting row i and column j of A.

For example,

If $A=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then $A_{12}=\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}$ and $A_{32}=\begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$

Let A be any $m\times m$ matrix. We may write A as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix}$$

To compute $|A|$, pick any value of i ($i=1,2,\cdots,m$) and compute $|A|$:

$$|A| = (-1)^{i+1}a_{i1}|A_{i1}| + (-1)^{i+2}a_{i2}|A_{i2}| + \cdots + (-1)^{i+m}a_{im}|A_{im}|$$

Example: Find $|A|$ for $A=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Solution:

$$\begin{aligned} |A| &= 1\times\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\times\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\times\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ &= 1\times(-3) - 2\times(-6) + 3\times(-3) \\ &= 0 \end{aligned}$$

Exercise: Find $|A|$ for $A=\begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -3 \\ -3 & 2 & -4 \end{bmatrix}$.

$|A| = -1$

Property: Invertibility Criterion

If A is a square matrix, then A has an inverse if and only if $\det(A) \neq 0$.

Example: Show that the matrix A has no inverse.

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 3 \\ 5 & 6 & 2 & 6 \\ 2 & 4 & 0 & 9 \end{bmatrix}$$

Solution: Since $\det(A) = 0$, then the matrix A has no inverse..

Definition of Nonsingular and Singular

A matrix is **nonsingular** if its rank equals both the number of rows and the number of columns. Otherwise, it is **singular**.

Theorem

(a) If A is nonsingular, there is unique nonsingular matrix A^{-1} such that $A^{-1}A = I = AA^{-1}$.

(b) If A is nonsingular and B is a matrix for which either $AB = I$ or $BA = I$, then $B = A^{-1}$.

(c) Only nonsingular matrices have inverse.

(d) If $|A| \neq 0$, then A is nonsingular.