Superconvergence of high order FEMs for eigenvalue problems with periodic boundary conditions

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Abstract

We study Adini’s elements for nonlinear Schrödinger equations (NLS) defined in a square box with periodic boundary conditions. First we transform the time-dependent NLS to a time-independent stationary state equation, which is a nonlinear eigenvalue problem (NEP). A predictor–corrector continuation method is exploited to trace solution curves of the NEP. We are concerned with energy levels and superfluid densities of the NLS. We analyze superconvergence of the Adini elements for the linear Schrödinger equation defined in the unit square. The optimal convergence rate \( O(\hbar^2) \) is obtained for quasiuniform elements. For uniform rectangular elements, the superconvergence \( O(\hbar^{\nu}) \) is obtained for the minimal eigenvalue, where \( \nu = 1 \) or \( \nu = 2 \). The theoretical analysis is confirmed by the numerical experiments. Other kinds of high order finite element methods (FEMs) and the superconvergence property are also investigated for the linear Schrödinger equation. Finally, the Adini elements-continuation method is exploited to compute energy levels and superfluid densities of a 2D Bose–Einstein condensates (BEC) in a periodic potential. Numerical results on the ground state as well as the first few excited-state solutions are reported.

1. Introduction

During the past decade, Bose–Einstein condensates (BEC) in an optical lattice has been opening up intriguing possibilities for the study of coherent matter wave in periodic potentials [1–8]. The lattice potential is formed by overlapping two perpendicular optical standing waves with the BEC, which can be expressed as [7,8]

\[
U(x, y) = U_0 \{\cos^2(\kappa x) + \cos^2(\kappa y) + 2e_1 \cdot e_2 \cos \phi \cos(\kappa x) \cos(\kappa y)\},
\]

(1.1)

Here \( U_0 \) denotes the maximum potential of a single standing wave, \( \kappa = 2\pi/\mu \) is the magnitude of the wave vector of the lattice beams, with \( \mu \) the wavelength generated by a near-infrared laser diode, and \( e_1 \) and \( e_2 \) are the polarization vectors of the horizontal and vertical standing wave laser fields, respectively. The potential depth \( U_0 \) can be expressed in units of the recoil energy \( E_R = \hbar^2 \kappa^2/2m \), where \( m \) is the mass of a single atom, e.g., rubidium. The variable \( \phi \) denotes the time-phase difference between the two standing wave laser fields [9]. The governing equation for the BEC in a 2D optical lattice may be described by the nonlinear Schrödinger equation (NLS) or the Gross–Pitaevskii equation (GPE) [10,11]

\[
i\psi_t = -\frac{1}{2} \nabla^2 \psi + V(x)\psi + U(x)\psi + \mu|\psi|^2\psi, \quad \psi > 0, \quad x \in \Omega \subset \mathbb{R}^2, \quad \psi(x, t) = 0, \quad x \in \partial \Omega, \quad t \geq 0.
\]

(1.2)

Here \( \psi = \psi(x, t) \) is macroscopic wave function of the BEC with state variable \( x = (x, y) \), \( V(x) \) the trapping potential, \( U(x) \) the lattice potential defined in (1.1), \( \mu \) a constant, and \( \Omega \) a bounded domain in \( \mathbb{R}^2 \) with piecewise smooth boundary \( \partial \Omega \). For convenience, we assume that \( \Omega \) is a square box in this paper.

In [12], Wu and Niu studied superfluidity of a BEC in an optical lattice in one-dimension. Specifically, the Landau–Zener tunnelling and dynamical instability were investigated therein. Note that superflow of a BEC in an optical lattice is represented by a Bloch wave, a plane wave with periodic modulation of the amplitude.
In this paper, we are concerned with energy levels and superfluid densities of a BEC in an optical lattice in two-dimension. The governing equation is
\[
\i \Psi_t = -\frac{1}{2} \Delta \Psi + \tilde{U}(x,y) \Psi + \mu |\Psi|^2 \Psi = 0 \quad \text{in} \quad \Omega = (-\pi, \pi)^2,
\]
\[
\Psi(x,y) = \Psi(x + 2\pi, y) = \Psi(x, y + 2\pi),
\]
where \( \tilde{U}(x,y) = v_1 \cos \left( \frac{x}{d} \right) + v_2 \cos \left( \frac{y}{d} \right) \) with \( d \) the distance between neighbor wells, \( v_i \) positive constants, \( i = 1, 2 \). We remark here that the trapping potential \( V(x) \) in (1.2) is omitted in (1.3). Moreover, the Dirichlet boundary conditions in (1.2) is replaced by the periodic boundary conditions in (1.3).

An important invariant of the NLS is the mass conservation constraint, or the normalization of the wave function
\[
\int_{\Omega} |\Psi(x,t)|^2 \, dx = 1, \quad t \geq 0.
\]

Various numerical methods have been proposed to study quantum behavior of the BEC, see e.g. [13–16]. By substituting
\[
\Psi(x,t) = e^{i \lambda t} u(x)
\]
into (1.3), we obtain the nonlinear eigenvalue problem
\[
F(u, \lambda) = -\frac{1}{2} \Delta u(x) - \lambda u(x) + \tilde{U}(x) u(x) + \mu |u|^2 u(x) = 0 \quad \text{in} \quad \Omega,
\]
\[
u(x,y) = u(x + 2\pi, y) = u(x, y + 2\pi).
\]

In general the function \( u(x) \) in (1.5) is a complex function. For simplicity we assume \( u(x) \) is a real function. Note that (1.6) is a parameter-dependent problem which can be solved using numerical continuation methods. For instance, Chien et al. proposed some variants of two-grid continuation schemes [17,18] to investigate energy levels and superfluid densities of rotating BEC [19].

The Schrödinger eigenvalue problem (SEP) associated with (1.6) is
\[
-\frac{1}{2} \Delta u(x) - \lambda u(x) + \tilde{U}(x) u(x) = 0 \quad \text{in} \quad \Omega = (-\pi, \pi)^2,
\]
\[
u(x,y) = u(x + 2\pi, y) = u(x, y + 2\pi).
\]

To compute an energy level of the GPE using numerical continuation methods, we may trace the corresponding solution curve branching from a bifurcation point on the trivial solution curve \( \{ (u, \lambda) = (0, \bar{\lambda}) | \lambda \in \mathbb{R} \} \), (1.6). We stop the curve-tracking whenever the constraint,
\[
\int_{\Omega} |u(x)|^2 \, dx = 1
\]
is satisfied. The constraint (1.8) is referred to as a target point on the solution curve. Note that a bifurcation point of the GPE is just an eigenvalue of the associated SEP [20], which can be computed using numerical methods. Our aim is to study high order finite element methods (FEM) for the GPE. In particular, the Adini element approximations will be incorporated in the context of continuation methods for curve-tracking. Of particular interest here is the investigation of superconvergence for the simplified SEP
\[
-\frac{1}{2} \Delta u = \lambda u \quad \text{in} \quad \Omega = (-\pi, \pi)^2,
\]
\[
u(x,y) = u(x + 2\pi, y) = u(x, y + 2\pi).
\]

For eigenvalue problems of self-adjoint elliptic of differential operators, the finite element method was studied in Strang and Fix [21] and in Babuška and Osborn [22], and the error estimates were derived. The eigenvalue problems for algebraic equations and linear operators were also reported in Wilkinson [23] and Chatelin [24], respectively. Global superconvergence was developed for elliptic problems in Chen and Huang [25], Lin and Yan [26] and Yan [27], and applied for eigenvalue problems in Lin and Lin [28] and Yang [29]. We will discuss periodic eigenvalue problems given in [30], and employ the direct constraints in Li [31] to deal with periodic boundary conditions. Moreover, high order FEMs, such as Adini’s elements, bi-quadratic and kth-order triangular elements, are chosen. We analyze superconvergence and derive new error bounds for high order FEMs.

In this paper we also give some numerical results of the Adini elements for the simplified SEP with different boundary conditions, i.e., Dirichlet, Neumann, periodic, and the Robin boundary conditions. Adini’s elements are defined on rectangles \( C_{0i} \) with nodal variables, \( u, u_0, u_4 \), at four corners of \( C_{0i} \), and the interpolant functions are expressed by polynomials of \( P_3 = P_1 + \text{span} \{ x^2 y, x y^2 \} \), where \( P_3 = \sum_{i,j} d_{ij} \delta_i \delta_j \) are cubic polynomials. Adini’s elements were first studied in Adini and Clough [32] and Melosh [33]. In some literature, Adini–Clough–Melosh rectangle is called, see [34]. For simplicity, we call it Adini’s elements in this paper. Error analysis of Adini’s elements was given in many papers for the fourth order elliptic problems, e.g., the biharmonic equations. We only mention some of them: Lascaux and Lesaint [35], Kikuchi [36], and Miyoshi [37] and Carlet [38].

From the numerical results we obtain the superclose \( |u_4 - u_0| = O(h^3) \) and \( |u_4 - u(0)| = O(h^3) \) for the uniform rectangles \( C_{0i} \), where \( u_0 \) is Adini’s interpolant based on the true solution \( u \) and \( u_4 \) the approximation solution for Adini’s elements. By the a posteriori interpolant we get the global superconvergence \( O(h^6) \) and \( O(h^8) \) in \( L_2 \) norm and \( H_1 \) norm, respectively. Furthermore, we also obtain the convergence rate \( O(h^8) \) of the minimal eigenvalue and the superconvergence \( O(h^8) \) by the Rayleigh quotient based on the a posteriori interpolant in Yang [29]. Moreover, we can obtain higher accuracy and convergence rates by the extrapolation formulas based on the approximate eigenvalues.

This paper is organized as follows. In the next section, the linear eigenvalue problem (LEP) with Dirichlet and Neumann boundary conditions involving periodic boundary conditions is described, which are denoted by Models I and II, respectively. In Section 3, the Adini elements are employed for the LEP with Models I and II. In Section 4, superconvergence of Adini’s elements by direct constraints is derived for Poisson’s equation with Models I and II. In Section 5, the superconvergence is developed for LEP. In Section 6, the superconvergence is applied to other FEMs for Models I and II. In Section 7, numerical results are reported to support our theoretical analysis. Moreover, the ground state solutions as well as the first few excited–state solution of (1.6) are presented. Finally, some concluding remarks are given in Section 8.

2. Periodic boundary conditions

For convenience we rewrite (1.9) as
\[
-\Delta u = \lambda u \quad \text{in} \quad S = (0, 1)^2,
\]
\[
u^+ = u^+ \quad u^{+0} = u_0^+ \quad \text{on} \quad \Gamma^+,
\]
where \( \Gamma = \partial S = \Gamma^+ \cup \Gamma^-, \Gamma^+ = \partial B \cup \partial D, \Gamma^- = \partial A \cup \partial C, \) and \( u^+ = u|_{\Gamma^+} \) (see Fig. 1). In (2.2), we impose the normal derivative \( u_n = u_0^+ \) on \( \Gamma \) or \( u_n = u_0^- \) on \( \partial S \) as shown in Fig. 1. Because of finite element approximations. The simplified notations in (2.2) denote
\[
u^+(x, 0) = u^-(x, 0), \quad u^+(y, 0) = u^-(0, y)
\]
and
\[
u^+(x, 1) = u_0^+(x, 0), \quad u^+(y, 1) = u_0^+(0, y).
\]

Obviously, \( \lambda = 0 \) is the minimal eigenvalue of (2.1) and (2.2) with corresponding eigenfunction, \( u \equiv \text{constant} \). We are interested in
the first few nonzero eigenvalues of (2.1) as well as their corresponding eigenfunctions.

Denote $\Gamma^+ = \overline{BD}$ and $\Gamma^- = \overline{AC}$ (see Fig. 2). For simplicity, we only consider the following mixed boundary conditions involving periodic boundary conditions.

I. The Dirichlet and Periodic boundary conditions

$$u(x,1) = u(x,0) = 0 \quad \text{on } AB \cup CD, \quad (2.5)$$

$$u^+ = u^- \quad \text{on } \Gamma^+, \quad (2.6)$$

which denote $u^+(1,y) = u^-(0,y)$ and $u^+_i(1,y) = u^-_i(0,y), \, 0 \leq y \leq 1$.

II. The Neumann and Periodic boundary conditions

$$u_x(x,1) = u_x(x,0) = 0 \quad \text{on } AB \cup CD, \quad (2.7)$$

$$u^+ = u^- \quad \text{on } \Gamma^+, \quad (2.8)$$

Eq.(2.1) with (2.5) and (2.6) (or with (2.7) and (2.8)) is called Model I (or Model II) in this paper.

For the boundary conditions (2.5) and (2.6), the minimal eigenvalue is positive. Denote $\hat{H}^1(S) = \{ v \in H^1(S), \, v \text{ satisfies (2.2)} \}$.

Eqs. (2.1) and (2.2) read: To seek $\lambda \in \mathbb{R}, \, 0 \neq u \in \hat{H}^1(S)$ such that

$$a(u,v) = \lambda b(u,v), \quad \forall v \in \hat{H}^1(S), \quad (2.11)$$

For simplicity, in this paper we only discuss Models I and II. The approaches and results of Model II can be applied easily to problems (2.1) and (2.2).

For Models I and II, denote

$$H^1(S) = \{ v \in H^1(S), \, v \text{ satisfies (2.5) and (2.6)} \} \quad (2.12)$$

and

$$H^1(S) = \{ v \in H^1(S), \, v \text{ satisfies (2.7) and (2.8) as well as (2.10)} \}, \quad (2.13)$$

respectively. Note that the constraint (2.10) is required for the Neumann problems by Adini's elements, to provide the unique solutions, see Huang et al. [38, 39]. The solution for Models I and II reads: To seek $\lambda \in \mathbb{R}, \, 0 \neq u \in H^1(S)$ such that

$$a(u,v) = \lambda b(u,v), \quad \forall v \in H^1(S), \quad (2.14)$$

where $a(u,v)$ and $b(u,v)$ are given in (2.9).

3. Adini’s elements for eigenvalue problems

Let S be divided by rectangular elements $\Box_y$ shown in Fig. 3. In each rectangular element, the $u_x$ and $u_y$ at four corners are chosen as unknown variables. The piecewise polynomial $v$ on $\Box_y$ is expressed by

$$v_h \in \hat{P}_3 = P_3(x,y) \oplus \text{span}\{x^2y, xy^2\}, \quad (3.1)$$

where $P_3(x,y) = \text{span}\{1, x, y, xy, x^2, y^2, x^3, y^3\}$. Hence, Adini's elements are three-order elements. Denote the finite dimensional space $V_h$ by

$$V_h = \{ v \in \hat{H}^1(S) \}, \quad (3.2)$$

where $\hat{H}^1(S)$ is given in (2.12) and (2.13) for Models I and II, respectively. The Adini elements for (2.14) read: To seek $\lambda_h \in \mathbb{R}, \, 0 \neq u_h \in V_h$ such that

$$a(u_h,v) = \lambda_h b(u_h,v), \quad \forall v \in V_h. \quad (2.10)$$

For problems (2.1) and (2.2) and Model II, we add a constraint on $u$ at one node, say at $A \in S$,

$$u(A) = u(0,0) = 0 \quad (2.10)$$
where \( a(u, v) \) and \( b(u, v) \) are given in (2.9).

Denote \( h_i \) and \( k_j \) the boundary lengths of \( \square_i \) (see Fig. 4). The rectangles \( \square_i \) are said to be quasiuniform if the following ratios are bounded,

\[
\frac{h_i}{\min\{h_i, k_j\}} \leq C, \tag{3.4}
\]

where \( h = \max_i\{h_i, k_j\} \) and \( C \) is a constant independent of \( h \). The rectangles \( \square_i \) are uniform if \( \square_i \) are quasiuniform and \( h_i = h \) and \( k_j = k \).

Denote \( \square_i = \{(x, y); x_i \leq x \leq x_{i+1}, y_j \leq y \leq y_{j+1}\} \). We choose the affine transformation: \( \xi = \frac{x-x_i}{h} \) and \( \eta = \frac{y-y_j}{k} \), where \( h_i = x_{i+1} - x_i \) and \( k_j = y_{j+1} - y_j \). Then the admissible functions are given by

\[
v(x, y) = \sum_{i=1}^{4} \psi_i(\xi, \eta) + \sum_{i=1}^{4} (v_i) \phi_i(\xi, \eta) + \sum_{i=1}^{4} (v_j) \phi_j(\xi, \eta), \tag{3.5}
\]

where the nodal points 1,2,3,4 denote \((i, j), (i+1, j), (i, j+1), (i+1, j+1)\), respectively. The 12 basis functions on \( [0, 1]^2 \) are given explicitly by [38]

\[
\phi_1(x, y) = (1-x)(1-y)(1+x+y-2x^2-2y^2), \\
\phi_2(x, y) = x(1-y)(2x+y-2x^2-2x^2), \\
\phi_3(x, y) = (1-x)y(x+5y-2x^2-2y^2), \\
\phi_4(x, y) = 2y(1-3y-x+3y-2x^2-2y^2), \\
\phi_5(x, y) = \psi_1(x, y) = (1-y)\bar{\Psi}_0(x), \\
\phi_6(x, y) = \psi_2(x, y) = (1-y)\bar{\Psi}_1(x), \\
\phi_7(x, y) = \psi_3(x, y) = y\bar{\Psi}_0(x), \\
\phi_8(x, y) = \psi_4(x, y) = y\bar{\Psi}_1(x), \\
\phi_9(x, y) = \theta_1(x, y) = (1-x)\bar{\Psi}_0(y), \\
\phi_{10}(x, y) = \theta_2(x, y) = x\bar{\Psi}_0(y), \\
\phi_{11}(x, y) = \theta_3(x, y) = (1-x)\bar{\Psi}_1(y), \\
\phi_{12}(x, y) = \theta_4(x, y) = x\bar{\Psi}_1(y), \tag{3.6}
\]

where the cubic-Hermite basis functions on \([0, 1]\) are

\[
\bar{\Psi}_0(s) = s^3 - 2s^2 + s, \quad \bar{\Psi}_1(s) = s^3 - s^2. \tag{3.7}
\]

It is well known that when the \( \square_i \) are quasiuniform and \( u \) is \( H^2(S) \), where \( H^2(S) \) is the Sobolev space with the Sobolev norm \( \|u\|_{H^2(S)} \), there exist the error bounds of Adini's elements for the leading eigenvalues \([34, 39, 21]\),

\[
|\lambda - \lambda_0| = O(h^6), \tag{3.8}
\]

and their corresponding eigenfunctions

\[
\|u - u_0\|_{1,5} = O(h^2). \tag{3.9}
\]

This section is devoted to achieve the superconvergence \( O(h^{6+p}) \) of the first few minimal eigenvalues, where \( p = 1 \) and \( p = 2 \) for Models I and II, respectively. For (2.5) and (2.6) in Model I, we use the direct constraints

\[
u_{i,M} = u_{i,0} = 0, \quad i = 0, 1, \ldots, N, \tag{3.10}
\]

\[
u_{i,j} = u_{i,0}j, \quad j = 0, 1, \ldots, M. \tag{3.11}
\]

For (2.7) and (2.8) in Model II, we use

\[
u_{i,j} = u_{i,j}, \quad j = 0, 1, \ldots, M. \tag{3.12}
\]

\[
u_{i,j} = u_{i,0}, \quad j = 0, 1, \ldots, M. \tag{3.13}
\]

\[
u_{i,j} = u_{0}, \tag{3.14}
\]

where Eq. (3.14) corresponds to (2.10).

Denote

\[
V_h = \{ v | v = v_h \in H^1(S), v_h \text{ satisfies (3.10) and (3.11) for Model I, or } v_h \text{ satisfies (3.12), (3.13) and (3.14) for Model II}, \}
\]

The Adini elements read: To seek \( \lambda_{\text{h}} \in R, 0 = u_{\text{h}} \in V_h \) such that

\[
a(u_{\text{h}}, v) = \lambda_{\text{h}} b(u_{\text{h}}, v), \quad \forall v \in V_h. \tag{3.15}
\]

Eqs. (3.10)-(3.14) are called the direct constraints of boundary conditions, and Eq. (3.16) is called the Adini elements (or Adini's elements) with direct constraints. We will show that when \( u \in H^2(S) \) and \( \square_i \) are chosen to be uniform, there exist the errors

\[
|\lambda - \lambda_{\text{h}}| = O(h^{6+p}), \tag{3.17}
\]

where \( q = 1 \) and \( q = 2 \) for Models I and II, respectively, and the leading eigenvalues \( \lambda_{\text{h}} \) are obtained by the Rayleigh quotient [29].

\[
\lambda_{\text{h}} = \frac{a(w, w)}{b(w, w)}, \quad w = \pi_{\text{h}}^S u_{\text{h}}. \tag{3.18}
\]

In (3.18) it the operation \( \pi_{\text{h}}^S \) is the a posteriori interpolant of order 4 based on the known \( u_{\text{h}} \) obtained from (3.16). The explicit formula for \( \pi_{\text{h}}^S \) can be found in [40].

Note that the constraints of normal derivatives in (3.12) for Model II are imperative to achieve the high superconvergence

\[
|\lambda - \lambda_{\text{h}}| = O(h^8). \tag{3.19}
\]

4. Error analysis for Poisson's equation

First, we consider Poisson's equation

\[-Au = f \text{ in } S. \tag{4.1}\]

with mixed type periodic boundary conditions in Models I and II, see (2.5), (2.6), and (2.7), (2.8) as well as (2.10). The weak solution of Poisson's equation is expressed as: to seek \( u \in H^1(S) \) such that

\[
a(u, v) = f(v), \quad \forall v \in H^1(S), \tag{4.2}
\]

where \( f(v) = \int_S f \cdot v \). For mixed boundary conditions (2.12) and (2.13), the Adini elements with direct constraints read: To seek \( u_{\text{h}} \in V_h \) such that

\[
a(u_{\text{h}}, v) = f(v), \quad \forall v \in V_h. \tag{4.3}
\]

where \( V_h \) is given in (3.2). The notation \( u_{\text{h}} \) of Poisson's solutions with a hat on the head is used to distinguish it from the solution \( u \) in (3.3).

If \( u \in H^2(S) \) and \( \square_i \) are uniform, there exist the superconvergence results [38]:

\[
\|u - u_{\text{h}}\|_{1,5} = O(h^{5}) \quad \text{or} \quad O(h^4), \tag{4.4}
\]

\[
\|u - \pi_{\text{h}}^S u_{\text{h}}\|_{1,5} = O(h^{5}) \quad \text{or} \quad O(h^4). \tag{4.5}
\]
for Model I (or Model II), where \( u_i \) is the Adini interpolant of \( u \), and \( \pi_p^1 \) is given in (3.18).

On \( \overline{\mathbb{AB} \cup \mathbb{CD}} \), the constraints (3.10) and (3.12) with (3.14) retain for the Dirichlet and the Neumann boundary conditions, respectively, but we replace the periodic boundary conditions on \( I^+ \) by the direct constraints in (3.11) and (3.13). Such Adini elements read: To seek \( \hat{u}_i \in V_h^p \) such that
\[
\hat{a}(\hat{u}_i, v) = f(v), \quad \forall v \in V_h^p. \tag{4.6}
\]

First we prove the following lemma.

**Lemma 4.1.** Let \( u \in H^2(\Omega) \) and \( \square \Omega \) be uniform, there exists the bound for all \( w \in V_h^p \),
\[
\hat{a}(u - \hat{u}_i, w) \leq Ch^{1-q} || u ||_{1,5} || w ||_h, \tag{4.7}
\]
where \( q = 0.5 \) and \( q = 1 \) for Models I and II, respectively, \( C \) is a constant independent of \( h \), and \( u_i \) is Adini’s interpolant of the true solution \( u \) for (4.1).

**Proof.** We have for the periodic boundary conditions on \( I^+ \) from Huang et al. [38],
\[
\hat{a}(u - \hat{u}_i, w) \leq Ch^{1} || u ||_{1,5} \left\{ \left( || w ||_{1,5} + \frac{|| \nabla w ||}{\alpha_{\mathbb{AB} \cup \mathbb{CD}}} \right) + \frac{|| \nabla w - \nabla \hat{w} ||}{\alpha_{\mathbb{AB} \cup \mathbb{CD}}} \right\}, \tag{4.8}
\]
where \( \hat{w} \) is the piecewise linear interpolant of \( w \) along \( \partial \Omega \). Under direct constraints in (3.11) and (3.13) we have \( \hat{w} = w \) on \( I^+ \) and for Model II \( \hat{w} = 0 \) on \( \overline{\mathbb{AB} \cup \mathbb{CD}} \). Eq. (4.8) leads to
\[
\hat{a}(u - \hat{u}_i, w) \leq Ch^{1} || u ||_{1,5} || w ||_{1,5}. \tag{4.9}
\]
For the Dirichlet boundary conditions in Model I, there exists the inverse inequality for the piecewise polynomials \( w \in V_h^p \),
\[
\left( \frac{|| \nabla w ||}{\alpha_{\mathbb{AB} \cup \mathbb{CD}}} \right) \leq Ch^{1} \left( || w ||_{1,5} \right), \quad \forall w \in V_h^p. \tag{4.10}
\]
From (4.8) and (4.10), we obtain
\[
\hat{a}(u - \hat{u}_i, w) \leq Ch^{1} || u ||_{1,5} || w ||_{1,5}. \tag{4.11}
\]
Hence the high superconvergence \( O(h^{1+q}) \) may be reached for Models I and II. This gives the desired result (4.7). □

We have the following lemma.

**Lemma 4.2.** There exist the uniformly \( V_h^p \) elliptic inequality
\[
c_o || v ||_{1,5} \leq a(v, v), \quad \forall v \in V_h^p \tag{4.12}
\]
and the bilinear inequality
\[
a(u, v) \leq C || u ||_{1,5} || v ||_{1,5}, \quad \forall v \in V_h^p, \tag{4.13}
\]
where \( c_o \) and \( C \) are two positive constants independent of \( h \).

Now, we give the main theorem of this section.

**Theorem 4.3.** Let \( u \in H^2(\Omega) \) and \( \square \Omega \) be uniform, there exist the error bounds
\[
|| \hat{u}_i - u ||_{1,5} \leq Ch^{3-q} || u ||_{1,5}, \tag{4.14}
\]
\[
|| u - \pi_p^1 u ||_{1,5} \leq Ch^{1-q}, \tag{4.15}
\]
where \( q = 0.5 \) and \( q = 1 \) for Models I and II, respectively, \( C \) is a constant independent of \( h \), and \( \pi_p^1 \) is the a posteriori interpolant of order 4 (see [40]).

**Proof.** For the true solution \( u \) of (4.1), from the direct constraints in (3.11) and (3.13) for the periodic boundary conditions, we have \( v^* = v \). Thus,
\[
\hat{a}(u, v) = f(v) + \int_{I^+} \frac{\partial \hat{a}}{\partial n} (v^* - v^*) = f(v), \quad \forall v \in V_h^p. \tag{4.16}
\]
Then from (4.6) and (4.16),
\[
\hat{a}(u - \hat{u}_i, v) = 0, \quad \forall v \in V_h^p. \tag{4.17}
\]
Denoting \( w = \hat{u}_i - u \in V_h^p \), we obtain from (4.17) and (4.12)
\[
c_o || w ||_{1,5} \leq \hat{a}(\hat{u}_i - u, w) = \hat{a}(u - u_i, w). \tag{4.18}
\]
From Lemma 4.1,
\[
\hat{a}(u - u_i, w) \leq Ch^{1-q} || u ||_{1,5} || w ||_{1,5}. \tag{4.19}
\]
where \( q = 0.5 \) and \( q = 1 \) for Models I and II, respectively. Eq. (4.14) follows by dividing both sides of (4.18) and (4.19) by \( || u_i - u ||_{1,5} \). Next, we have
\[
|| u - \pi_p^1 u ||_{1,5} \leq || u - \pi_p^1 u_i ||_{1,5} + || \pi_p^1 (u_i - u) ||_{1,5} \leq Ch^{1} || u_i ||_{1,5} + C || u_i - u ||_{1,5}. \tag{4.20}
\]
where we have used the bound \( || \pi_p^1 u ||_{1,5} \leq C || v ||_{1,5} \) for all \( v \in V_h^p \). Combining (4.20) and (4.14) gives (4.15). This completes the proof of Theorem 4.3. □

Since \( || v ||_{1,5} \) is equivalent to \( || v ||_{0,5} \) from Theorem 4.3 we have
\[
\hat{u}_i - u \in O(h^{1+q}), \quad || u - \pi_p^1 u ||_{0,5} = O(h^{1+q}). \tag{4.21}
\]

5. Superconvergence for eigenvalue problems

In this section, we focus on the error analysis for Adini’s elements with periodic boundary conditions by the direct constraints in (3.11) and (3.13).

Let \( f = 0 \). Suppose that \( u^* \in H_0^1(\Omega) \) which satisfies
\[
a(u^*, v) = b(u^*, v), \quad \forall v \in H_0^1(\Omega) \tag{5.1}
\]
and \( u_i \in V_h^p(\Omega) \) which satisfies
\[
a(u_i, v) = b(u_i, v), \quad \forall v \in V_h^p(\Omega). \tag{5.2}
\]
Eqs. (5.1) and (5.2) can be regarded as Poisson’s equation,
\[
a(u^*, v) = f_0(v), \quad \forall v \in H_0^1(\Omega), \tag{5.3}
\]
a(u_i, v) = f_0(v), \quad \forall v \in V_h^p(\Omega), \tag{5.4}
where \( f_0(v) = \int f \psi v \). Now we give the main theorem of this section.

**Theorem 5.1.** Let \( u \in H^2(\Omega) \) and \( \square \Omega \) be quasuniform. There exists the bound,
\[
|| u_i^* - u ||_{1,5} \leq O(h^{3+q}), \tag{5.5}
\]
where \( u_i^* \) is given in (4.4), and \( q = 0.5 \) and \( q = 1 \) for Models I and II, respectively.

**Proof.** We have
\[
|| u_i^* - u ||_{1,5} \leq || u_i^* - u_i ||_{1,5} + || u_i - u_i ||_{1,5}. \tag{5.6}
\]
Since \( u_i \) in (5.2) is the Ritz projection of \( u = u^* \) in (5.1), we obtain from Theorem 4.3,
\[
|| u_i^* - u_i ||_{1,5} = || u_i^* - u_i ||_{1,5} = O(h^{3+q}). \tag{5.7}
\]

\[2\] In general, we may define \( b(u, v) = \int f_0(v) \psi v \) with the positive weight \( \rho > 0 \). Hence the results of this paper may be extended to the eigenvalues problem \( -\Delta u = \mu u \). For the case of this paper \( \rho = 1 \), \( || v ||_{1,5} = || v ||_0. \]
The results for (7.1), where Table 1
The desired result (5.5) follows from (5.6), (5.7) and (5.8). This completes the proof of Theorem 5.1. □

From Yan [27] (Lemma 4.6.1 in p. 171), there exists the bound
\[ \|u_h - u\|_{1,5} = O(h^{1+\eta}). \]  
(5.8)

The desired result (5.5) follows from (5.6), (5.7) and (5.8). This completes the proof of Theorem 5.1. □

Table 1

<table>
<thead>
<tr>
<th>N</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_h )</td>
<td>19.88176127611392</td>
<td>19.74266715086334</td>
<td>19.73927351049494</td>
<td>19.73920984531915</td>
</tr>
<tr>
<td>( \lambda_h - \lambda )</td>
<td>0.143</td>
<td>0.346(-2)</td>
<td>0.647(-4)</td>
<td>0.104(-5)</td>
</tr>
<tr>
<td>( |u - u_h|_{1,5} )</td>
<td>0.375(-1)</td>
<td>0.941(-3)</td>
<td>0.499(-5)</td>
<td>0.212(-7)</td>
</tr>
<tr>
<td>( u - u_h )</td>
<td>0.170(-1)</td>
<td>0.165(-2)</td>
<td>0.125(-3)</td>
<td>0.826(-5)</td>
</tr>
<tr>
<td>( |u - u_h|_{1,5} )</td>
<td>0.199</td>
<td>0.302(-1)</td>
<td>0.406(-2)</td>
<td>0.517(-3)</td>
</tr>
<tr>
<td>( u - u_h )</td>
<td>0.200</td>
<td>0.303(-1)</td>
<td>0.406(-2)</td>
<td>0.517(-3)</td>
</tr>
<tr>
<td>( |u - u_h|_{1,5} )</td>
<td>0.145(-1)</td>
<td>0.586(-3)</td>
<td>0.160(-4)</td>
<td>0.499(-6)</td>
</tr>
<tr>
<td>( u - u_h )</td>
<td>0.103</td>
<td>0.104(-1)</td>
<td>0.760(-3)</td>
<td>0.496(-4)</td>
</tr>
<tr>
<td>( |u - u_h|_{1,5} )</td>
<td>0.104</td>
<td>0.104(-1)</td>
<td>0.761(-3)</td>
<td>0.496(-4)</td>
</tr>
<tr>
<td>( u - u_h )</td>
<td>0.166(-1)</td>
<td>0.110(-2)</td>
<td>0.361(-4)</td>
<td>0.113(-5)</td>
</tr>
<tr>
<td>( |u - u_h|_{1,5} )</td>
<td>0.124</td>
<td>0.161(-1)</td>
<td>0.113(-2)</td>
<td>0.729(-4)</td>
</tr>
<tr>
<td>( u - u_h )</td>
<td>0.125</td>
<td>0.162(-1)</td>
<td>0.113(-2)</td>
<td>0.729(-4)</td>
</tr>
</tbody>
</table>

Note that the convergence rate in Theorem 5.1 is analogous to that in Theorem 4.3.

Lemma 5.2. For any \( w \in H^1(\Omega) \), there exists the bound
\[ \frac{a(w, w)}{\|w\|_b^2} - \lambda = \frac{a(w - u, w - u)}{\|w - u\|_b^2} - \lambda \frac{\|w - u\|_b^2}{\|w\|_b^2}. \]  
(5.9)

where \( \langle \cdot, \cdot \rangle \) is the eigenpair of (2.14), and \( \|w\|_b = \sqrt{b(w, w)} \).

Proof. For the true eigenpair \( (\lambda, u) \), we have from \( w^+ = w^- \),
\[ a(u, w) = \lambda b(u, w) + \int_{\Omega} \frac{\partial u}{\partial n} w = \lambda b(u, w). \]  
(5.10)

Thus \( a(u, w) = \lambda b(u, w) \).

Consider
\[ a(w - u, w - u) - \lambda b(w - u, w - u) = a(w, w) - 2a(w, u) + a(u, u) - \lambda \{b(w, w) - 2b(w, u) + b(u, u)\}. \]  
(5.11)

We have from (5.10) and (5.11)
\[ a(w - u, w - u) - \lambda \|w - u\|_b^2 = a(w, w) - \lambda b(w, w). \]  
(5.12)
Dividing the two sides of (5.12) by \(\|w\|_0^2\) gives the desired result (5.9). \(\square\)

We have the following result.

**Theorem 5.3.** Let all conditions in Theorem 5.1 hold. For the Rayleigh quotient \(\lambda_{h,p} = \frac{B(u,w)}{B(w,w)}\), \(w = \pi^h_p u_i\), there exists the bound for the leading eigenvalue

\[
|\lambda - \lambda_{h,p}| = O(h^{2q}),
\]

where \(q = 0.5\) and \(q = 1\) for Models I and II, respectively.

**Proof.** Construct the a posteriori interpolant \(\pi^l_p u_i\) of \(u_i\). From Theorem 5.1, we have

\[
\|u - \pi^l_p u_i\|_{1.5} \leq \|u - \pi^h_p u_i\|_{1.5} + \|\pi^l_p(u_i - u_h)\|_{1.5} \\
\leq C h^3 + C \|u_i - u_h\|_{1.5} \leq C (h^4 + h^{3-q}) \\
= C h^{3-q},
\]

where we have used that \(\|\pi^h_p v\|_{1.5} \leq C \|v\|_{1.5}\), \(\forall v \in V_i\). Also

\[
\|u - \pi^h_p u_i\|_h \leq C \|u - \pi^h_p u_i\|_{1.5} \leq C \|u - \pi^l_p u_i\|_{1.5} \leq C h^{3-q}.
\]

Choose \(w = \pi^l_p u_i\) with \(\|w\|_h \leq C \|u_i\|_{1.5} = C\). Since \(a(w,w) \leq C\|w\|_h^2\), from Lemma 5.2, (5.14) and (5.15) we have

\[
|\lambda_{h,p} - \lambda| \leq C (\|u - \pi^l_p u_i\|_{1.5} + \|u - \pi^h_p u_i\|_h^2) \leq C h^{6-2q}.
\]

This completes the proof of Theorem 5.3. \(\square\)

From Theorem 5.3, since \(q = 0.5\) and \(q = 1\) for Models I and II, we obtain the superconvergence, respectively,

\[
|\lambda_{h,p} - \lambda| = O(h^7)
\]

and

\[
|\lambda_{h,p} - \lambda| = O(h^8).
\]

Compared with the optimal convergence rates in (3.8), the higher orders such as \(O(h)\) and \(O(h^7)\) are achieved for Models I and II, respectively.

### 6. Applications to other kinds of FEMs

#### 6.1. Bi-quadratic Lagrange elements

Let us use the bi-quadratic Lagrange elements with point-line-area variables on \(\square\) as in [39,26]. Let \(v_i \in P_2(x,y) = \text{span}(1,x,y,x^2,y^2,x^2y,xy^2)\) which satisfies
Table 4
The extrapolation solutions for the Rayleigh quotient solutions of (7.1).

<table>
<thead>
<tr>
<th>$N$</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_{HP}$</td>
<td>19.77673671916215</td>
<td>19.74014932571034</td>
<td>19.73921378894563</td>
<td>19.73920882333811</td>
</tr>
<tr>
<td>$\lambda^{(1)}_{HP}$</td>
<td></td>
<td>19.74000584573602</td>
<td>19.73921012017400</td>
<td>19.73920880386514</td>
</tr>
<tr>
<td>$\lambda^{(2)}_{HP}$</td>
<td></td>
<td></td>
<td>19.73920934233865</td>
<td>19.7392088257842</td>
</tr>
<tr>
<td>$\lambda^{(3)}_{HP}$</td>
<td></td>
<td></td>
<td></td>
<td>19.7392088244661</td>
</tr>
<tr>
<td>$\lambda_{HP} - \lambda$</td>
<td>0.375(−1)</td>
<td>0.941(−3)</td>
<td>0.499(−5)</td>
<td>0.212(−7)</td>
</tr>
<tr>
<td>$\lambda^{(1)}_{HP} - \lambda$</td>
<td></td>
<td>0.797(−3)</td>
<td>0.132(−5)</td>
<td>0.169(−8)</td>
</tr>
<tr>
<td>$\lambda^{(2)}_{HP} - \lambda$</td>
<td></td>
<td></td>
<td>0.540(−6)</td>
<td>0.400(−9)</td>
</tr>
<tr>
<td>$\lambda^{(3)}_{HP} - \lambda$</td>
<td></td>
<td></td>
<td></td>
<td>0.268(−9)</td>
</tr>
<tr>
<td>$|u - u_{0}|_{0,5}$</td>
<td></td>
<td>39.9</td>
<td>189</td>
<td>236</td>
</tr>
<tr>
<td>$|u - u_{1,5}|_{0,5}$</td>
<td></td>
<td></td>
<td>605</td>
<td>782</td>
</tr>
<tr>
<td>$|u - u_{2,5}|_{0,5}$</td>
<td></td>
<td></td>
<td></td>
<td>1350</td>
</tr>
<tr>
<td>$|u - u_{3,5}|_{0,5}$</td>
<td></td>
<td></td>
<td></td>
<td>6</td>
</tr>
<tr>
<td>$|u - u_{4,5}|_{0,5}$</td>
<td></td>
<td></td>
<td></td>
<td>12</td>
</tr>
<tr>
<td>$|u - u_{5,5}|_{0,5}$</td>
<td></td>
<td></td>
<td></td>
<td>24</td>
</tr>
<tr>
<td>$|u - u_{6,5}|_{0,5}$</td>
<td></td>
<td></td>
<td></td>
<td>6</td>
</tr>
<tr>
<td>$|u - u_{7,5}|_{0,5}$</td>
<td></td>
<td></td>
<td></td>
<td>24</td>
</tr>
<tr>
<td>$|u - u_{8,5}|_{0,5}$</td>
<td></td>
<td></td>
<td></td>
<td>12</td>
</tr>
<tr>
<td>$|u - u_{9,5}|_{0,5}$</td>
<td></td>
<td></td>
<td></td>
<td>6</td>
</tr>
</tbody>
</table>

Table 5
The numerical results for (7.10).

<table>
<thead>
<tr>
<th>$N$</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_{H}$</td>
<td>9.872176668300702</td>
<td>9.869667076705241</td>
<td>9.869605535424065</td>
<td>9.869604419537595</td>
</tr>
<tr>
<td>$\lambda - \lambda$</td>
<td>0.257(−2)</td>
<td>0.627(−4)</td>
<td>0.113(−5)</td>
<td>0.184(−7)</td>
</tr>
<tr>
<td>$\lambda_{HP}$</td>
<td>9.871348857193135</td>
<td>9.869623410781877</td>
<td>9.869604491152328</td>
<td>9.869604401459600</td>
</tr>
<tr>
<td>$\lambda_{HP} - \lambda$</td>
<td>0.174(−2)</td>
<td>0.190(−4)</td>
<td>0.901(−7)</td>
<td>0.370(−9)</td>
</tr>
<tr>
<td>$|u - u_{0}|_{1,5}$</td>
<td>0.404(−2)</td>
<td>0.374(−3)</td>
<td>0.266(−4)</td>
<td>0.172(−5)</td>
</tr>
<tr>
<td>$|u - u_{1,5}|_{1,5}$</td>
<td>0.379(−1)</td>
<td>0.572(−2)</td>
<td>0.758(−3)</td>
<td>0.962(−4)</td>
</tr>
<tr>
<td>$|u - u_{2,5}|_{1,5}$</td>
<td>0.381(−1)</td>
<td>0.573(−2)</td>
<td>0.758(−3)</td>
<td>0.962(−4)</td>
</tr>
<tr>
<td>$|u - u_{3,5}|_{1,5}$</td>
<td>0.344(−2)</td>
<td>0.107(−3)</td>
<td>0.306(−4)</td>
<td>0.919(−7)</td>
</tr>
<tr>
<td>$|u - u_{4,5}|_{1,5}$</td>
<td>0.258(−1)</td>
<td>0.212(−2)</td>
<td>0.145(−3)</td>
<td>0.929(−5)</td>
</tr>
<tr>
<td>$|u - u_{5,5}|_{1,5}$</td>
<td>0.260(−1)</td>
<td>0.212(−2)</td>
<td>0.145(−3)</td>
<td>0.930(−5)</td>
</tr>
<tr>
<td>$|u - u_{6,5}|_{1,5}$</td>
<td>0.448(−2)</td>
<td>0.209(−3)</td>
<td>0.671(−4)</td>
<td>0.211(−6)</td>
</tr>
<tr>
<td>$|u - u_{7,5}|_{1,5}$</td>
<td>0.328(−1)</td>
<td>0.315(−2)</td>
<td>0.213(−3)</td>
<td>0.136(−4)</td>
</tr>
<tr>
<td>$|u - u_{8,5}|_{1,5}$</td>
<td>0.331(−1)</td>
<td>0.316(−2)</td>
<td>0.213(−3)</td>
<td>0.136(−4)</td>
</tr>
<tr>
<td>$|u - u_{9,5}|_{1,5}$</td>
<td></td>
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</tr>
<tr>
<td>$|u - u_{10,5}|_{1,5}$</td>
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</tr>
<tr>
<td>$|u - u_{11,5}|_{1,5}$</td>
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</tr>
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<tr>
<td>$|u - u_{16,5}|_{1,5}$</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

$\|u - u_{0}\|_{0,5}$ | 0.404(−2) | 0.374(−3) | 0.266(−4) | 0.172(−5) |
| $\|u - u_{1,5}\|_{0,5}$ | 0.379(−1) | 0.572(−2) | 0.758(−3) | 0.962(−4) |
| $\|u - u_{2,5}\|_{0,5}$ | 0.381(−1) | 0.573(−2) | 0.758(−3) | 0.962(−4) |
| $\|u - u_{3,5}\|_{0,5}$ | 0.344(−2) | 0.107(−3) | 0.306(−4) | 0.919(−7) |
| $\|u - u_{4,5}\|_{0,5}$ | 0.258(−1) | 0.212(−2) | 0.145(−3) | 0.929(−5) |
| $\|u - u_{5,5}\|_{0,5}$ | 0.260(−1) | 0.212(−2) | 0.145(−3) | 0.930(−5) |
| $\|u - u_{6,5}\|_{0,5}$ | 0.448(−2) | 0.209(−3) | 0.671(−4) | 0.211(−6) |
| $\|u - u_{7,5}\|_{0,5}$ | 0.328(−1) | 0.315(−2) | 0.213(−3) | 0.136(−4) |
| $\|u - u_{8,5}\|_{0,5}$ | 0.331(−1) | 0.316(−2) | 0.213(−3) | 0.136(−4) |
The ratios of error norms at $h$ and $2h$ for (7.10), based on Table 5.

<table>
<thead>
<tr>
<th>$\frac{e_h}{e_{2h}}$</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{e_{h}}{e_{2h}}$</td>
<td>41.0</td>
<td>55.3</td>
<td>59.7</td>
<td>61.5</td>
<td>62.9</td>
</tr>
<tr>
<td>$\frac{e_{h}}{e_{2h}}$</td>
<td>91.8</td>
<td>211</td>
<td>234</td>
<td>243</td>
<td>250</td>
</tr>
</tbody>
</table>

The FEM (4.3) retains the same, where $u_h$ is the bi-quadratic solution. Hence Theorems 5.1 and 5.3 hold, where $q = 1$ for both Models I and II.

6.2. Triangular elements

Consider $S$ as a parallelogram shown in Fig. 5. Let $S$ be divided into small parallelograms, and then the triangles are obtained by splitting the parallelograms with diagonals. For simplicity, we choose the linear and quadratic elements in [30], see Fig. 6. The a posteriori interpolants $\pi_{h_0}^k u_h$ and $\pi_{h_p}^k u_h$ of linear and quadratic elements are formulated based on the solutions at the elements nodes in Fig. 7.

The periodic boundary conditions (2.2), (2.6) and (2.8) are the same, where $I^+$ denote the same sections. Since $S$ is not rectangular, using the triangular elements is necessary. For Model II, condition (3.12) is not required in $V_h$. The $k$-order FEM using the direct constraints is given in (3.16), where $k = 1$ and $k = 2$ denote the linear and quadratic elements, respectively (see Fig. 7).

Similar the approaches in Sections 4 and 5, we can prove the following error bounds,

$$
\|u_h - u\|_{1,5} \leq Ch^{k+q} \|u\|_{k+q+1.5},
$$

(6.2)

where $u_h$ is the solution from (4.6), and $q = 1$. Hence we have

$$
\|u_h - u\|_{1,5} = O(h^{k+1}), 
$$

(6.3)

Denote ($\lambda_h^*, u_h^*$) as the FEM solution for (3.16) and

$$
\lambda_h^* = \frac{a(w, w)}{\|w\|_b^2}, \quad w = \pi_{h_p}^{k-1} u_h.
$$

(6.4)

If $S = \cup_k \Delta_k$, and $\Delta_k$ are uniform, we can also prove the error bounds

$$
|\lambda_h^* - \lambda| = O(h^{2k+2}), 
$$

(6.5)

The numerical results for (7.15).

| $\lambda_h$ | 9.872176668300702 | 9.8869607076705241 | 9.869605535424065 | 9.869604419537595 |
| $\lambda_h - \lambda$ | 0.257 (-2) | 0.627 (-4) | 0.113 (-5) | 0.184 (-7) |
| $\lambda_h$ | 9.87828441912027 | 9.869623410781876 | 9.86960491152334 | 9.869604014559593 |
| $\lambda_h - \lambda$ | 0.322 (-2) | 0.190 (-4) | 0.901 (-7) | 0.370 (-9) |
| $\|u - u_h\|_{0,5}$ | 0.404 (-2) | 0.374 (-3) | 0.266 (-4) | 0.172 (-5) |
| $\|u - u_h\|_{1,5}$ | 0.379 (-1) | 0.572 (-2) | 0.758 (-3) | 0.962 (-4) |
| $\|u - u_h\|_{1,5}$ | 0.381 (-1) | 0.573 (-2) | 0.758 (-3) | 0.962 (-4) |
| $\|u - u_h\|_{1,5}$ | 0.344 (-2) | 0.107 (-3) | 0.306 (-5) | 0.191 (-7) |
| $\|u - u_h\|_{1,5}$ | 0.258 (-1) | 0.212 (-2) | 0.145 (-3) | 0.929 (-5) |
| $\|u - u_h\|_{1,5}$ | 0.260 (-1) | 0.212 (-2) | 0.145 (-3) | 0.930 (-5) |
| $\|u - \pi_{h_p}^{k-1} u_h\|_{0,5}$ | 0.697 (-2) | 0.209 (-3) | 0.671 (-5) | 0.211 (-6) |
| $\|u - \pi_{h_p}^{k-1} u_h\|_{1,5}$ | 0.460 (-1) | 0.315 (-2) | 0.213 (-3) | 0.136 (-4) |
| $\|u - \pi_{h_p}^{k-1} u_h\|_{1,5}$ | 0.465 (-1) | 0.316 (-2) | 0.213 (-3) | 0.136 (-4) |

| 6 | 12 | 24 |
| $\lambda_h$ | 9.869610524378438 | 9.869604503658393 | 9.86960402721281 |
| $\lambda_h - \lambda$ | 0.612 (-5) | 0.103 (-6) | 0.163 (-8) |
| $\lambda_h$ | 9.869605255417273 | 9.869604407384744 | 9.86960401103931 |
| $\lambda_h - \lambda$ | 0.854 (-6) | 0.365 (-8) | 0.146 (-10) |
| $\|u - u_h\|_{0,5}$ | 0.812 (-4) | 0.540 (-5) | 0.343 (-6) |
| $\|u - u_h\|_{1,5}$ | 0.177 (-2) | 0.227 (-3) | 0.286 (-4) |
| $\|u - u_h\|_{1,5}$ | 0.177 (-2) | 0.227 (-3) | 0.286 (-4) |
| $\|u - u_h\|_{1,5}$ | 0.133 (-4) | 0.392 (-6) | 0.120 (-7) |
| $\|u - u_h\|_{1,5}$ | 0.447 (-3) | 0.292 (-4) | 0.184 (-5) |
| $\|u - u_h\|_{1,5}$ | 0.447 (-3) | 0.292 (-4) | 0.184 (-5) |
| $\|u - \pi_{h_p}^{k-1} u_h\|_{0,5}$ | 0.281 (-4) | 0.887 (-6) | 0.278 (-7) |
| $\|u - \pi_{h_p}^{k-1} u_h\|_{1,5}$ | 0.660 (-3) | 0.428 (-4) | 0.270 (-5) |
| $\|u - \pi_{h_p}^{k-1} u_h\|_{1,5}$ | 0.660 (-3) | 0.428 (-4) | 0.270 (-5) |
Remark 6.1. Since a pair of triangles forms a parallelogram, the triangulation is strongly regular. The superconvergence can be found in [41, 26, 28, 27]. For $k = 1$ and $k = 2$, we obtain from (6.5)

$$|\lambda_{h,p} - \lambda| = O(h^4) \tag{6.6}$$

for linear elements, and

$$|\lambda_{h,p} - \lambda| = O(h^6)$$

for quadratic elements. In [30], only the $O(h^4)$ is obtained from quadratic elements; such a convergence rate can be achieved by linear elements by superconvergence (see (6.6)). Moreover, the $O(h^6)$ can be reached by the quadratic elements using the techniques in this paper.

7. Numerical results

In this section, we give some numerical results for the eigenvalue problems with different boundary conditions, such as the Dirichlet boundary conditions, the Neumann boundary conditions.

**Table 8**
The ratios of error norms at $h$ and $2h$ for (7.15), based on Table 9.

<table>
<thead>
<tr>
<th>$N$</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{|u|<em>{\infty,1}^2}{|u|</em>{\infty,2}^2}$</td>
<td>41.0</td>
<td>55.3</td>
<td>59.7</td>
<td>61.5</td>
<td>62.9</td>
</tr>
<tr>
<td>$\frac{|u|<em>{\infty,1}}{|u|</em>{\infty,2}}$</td>
<td>170</td>
<td>211</td>
<td>234</td>
<td>243</td>
<td>250</td>
</tr>
<tr>
<td>$\frac{|u|<em>{1,1}^2}{|u|</em>{1,2}^2}$</td>
<td>10.8</td>
<td>14.0</td>
<td>15.0</td>
<td>15.4</td>
<td>15.7</td>
</tr>
<tr>
<td>$\frac{|u|<em>{1,1}}{|u|</em>{1,2}}$</td>
<td>6.63</td>
<td>7.55</td>
<td>7.79</td>
<td>7.88</td>
<td>7.94</td>
</tr>
<tr>
<td>$\frac{|u|<em>{p,1}^2}{|u|</em>{p,2}^2}$</td>
<td>6.65</td>
<td>7.56</td>
<td>7.79</td>
<td>7.88</td>
<td>7.95</td>
</tr>
<tr>
<td>$\frac{|u|<em>{p,1}}{|u|</em>{p,2}}$</td>
<td>32.2</td>
<td>34.9</td>
<td>34.0</td>
<td>33.3</td>
<td>32.7</td>
</tr>
<tr>
<td>$\frac{|u|<em>{p,0}^2}{|u|</em>{p,0}}$</td>
<td>12.2</td>
<td>14.6</td>
<td>15.3</td>
<td>15.6</td>
<td>15.8</td>
</tr>
<tr>
<td>$\frac{|u|<em>{p,0}}{|u|</em>{p,0}}$</td>
<td>12.3</td>
<td>14.6</td>
<td>15.3</td>
<td>15.6</td>
<td>15.8</td>
</tr>
<tr>
<td>$\frac{|u|<em>{p,1}^2}{|u|</em>{p,0}}$</td>
<td>33.3</td>
<td>31.2</td>
<td>31.7</td>
<td>31.8</td>
<td>31.9</td>
</tr>
<tr>
<td>$\frac{|u|<em>{p,1}}{|u|</em>{p,0}}$</td>
<td>14.6</td>
<td>14.8</td>
<td>15.4</td>
<td>15.7</td>
<td>15.8</td>
</tr>
<tr>
<td>$\frac{|u|<em>{p,2}^2}{|u|</em>{p,0}}$</td>
<td>14.7</td>
<td>14.8</td>
<td>15.4</td>
<td>15.7</td>
<td>15.8</td>
</tr>
</tbody>
</table>

**Table 9**
The results for (7.16), where $h - \frac{1}{4}$ and $\lambda_i$ are the $i$th eigenvalue.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\lambda_3$</th>
<th>$\lambda_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>10.205394884</td>
<td>40.3094552430</td>
<td>50.2148437157</td>
<td>52.3211430368</td>
</tr>
<tr>
<td>4</td>
<td>10.2029768990</td>
<td>39.8222198478</td>
<td>49.7393188543</td>
<td>49.7393188543</td>
</tr>
<tr>
<td>8</td>
<td>10.2029294646</td>
<td>39.8120115921</td>
<td>49.6827804588</td>
<td>49.6827804588</td>
</tr>
<tr>
<td>16</td>
<td>10.2029351347</td>
<td>39.8117542503</td>
<td>49.6813817395</td>
<td>49.6813817395</td>
</tr>
<tr>
<td>24</td>
<td>10.2029365493</td>
<td>39.8117504444</td>
<td>49.6813572388</td>
<td>49.6813572388</td>
</tr>
</tbody>
</table>

**Fig. 8.** The contours of $u_i(x)$ at the target points $(\|u_i\|_{\infty,1}^2, \lambda_i^2)$ of the (a) first, (b) second, (c) third, and (d) fourth solution branches of Eq. (1.6) with $v_1 = v_2 = 100$, $\mu = 10$, and $d_1 = d_2 = 1/2$. 

and periodic boundary conditions. The uniform meshsize for the Adini elements is $h = \frac{1}{N}$. In particular, the first few energy levels and superfluid densities of a 2D BEC in a periodic potential are presented.

**Example 1.** We used the Adini elements to discretize the simplified Schrödinger eigenvalue problem with Dirichlet boundary conditions

$$(-\Delta u + \lambda u = 0 \quad \text{in} \quad S = [0, 1]^2, \quad u = 0 \quad \text{on} \quad \partial S.)$$

(7.1)

The eigenpairs of (7.1) are

$$u_{k,\ell} = \sin(k\pi x) \sin(\ell\pi y), \quad \lambda_{k,\ell} = (k^2 + \ell^2)\pi^2, \quad 1 \leq k, \ell \leq N - 1.$$  

The numerical results are listed in Table 1. Based on the results in Table 1 we list the ratio of errors in Table 2. From Tables 1 and 2, we can see the following asymptotic convergence rates for the minimal eigenvalue $\lambda_1 = 2\pi^2$ and the eigenfunction corresponding to $\lambda_1$.

$$|\lambda_h - \lambda_1| = O(h^8), \quad |\lambda_{h,p} - \lambda_1| = O(h^8),$$

(7.2)

$$\|u - u_h\|_{L^2} = O(h^3), \quad \|u - u_{h,p}\|_{L^2} = O(h^3),$$

(7.3)

$$\|u - u_h\|_{L^2} = O(h^3), \quad \|u - u_{h,p}\|_{L^2} = O(h^3),$$

(7.4)

$$\|u - \pi_h^u u_h\|_{L^2} = O(h^3), \quad \|u - \pi_h^u u_{h,p}\|_{L^2} = O(h^3).$$

(7.5)

where $u_h = u_n$, and $\lambda_{h,p}$ is obtained by the Rayleigh quotient (3.18): $\lambda_{h,p} = \frac{\langle w, u_h \rangle}{\langle u_h, w \rangle}$, $w = \pi_h^u u_h$. Note that $|\lambda_{h,p} - \lambda_1| = O(h^6)$ in (7.2) is of high convergence rates. The numerical results of superclose $\|u - u_h\|_{L^2} = O(h^3)$ and $\|u - u_{h,p}\|_{L^2} = O(h^3)$ in (7.4) are one order higher than those of the optimal rates $O(h^4)$ and $O(h^4)$ in [34]. Moreover, we use the a posteriori interpolant to obtain the superconvergence $O(h^4)$ and $O(h^4)$ in the $L_2$ norm and $H_1$ norm in (7.5), respectively.

Using the results in Table 1, we obtain more accurate eigenvalue approximations by the following extrapolation formulas,

$$\lambda_h^{(i)} = \frac{2^{6+2(i-1)}\lambda_h^{(i-1)} - \lambda_h^{(i-2)}}{2^{6+2(i-1)} - 1}, \quad i = 1, 2, 3,$$

(6.7)

$$\lambda_{h,p}^{(i)} = \frac{2^{6+2(i-1)}\lambda_{h,p}^{(i-1)} - \lambda_{h,p}^{(i-2)}}{2^{6+2(i-1)} - 1}, \quad i = 1, 2, 3,$$

(6.8)

where $u_h = u_0$, $\lambda_0 = \lambda_0$, $\lambda_{h,p} = \lambda_{h,p}$, and $\lambda_{h,p}^{(i)}$ and $\lambda_{h,p}^{(i)}$ are called the $i$th extrapolation of $\lambda_h$ and $\lambda_{h,p}$, respectively. The extrapolation results are listed in Tables 3 and 4. From Table 3, we see the following asymptotic convergence rates for $\lambda_h$,

$$\left| \frac{\lambda_h^{(1)} - \lambda}{\lambda_h^{(2)} - \lambda} \right| = 213 \approx 2^{7.7}, \quad \left| \frac{\lambda_h^{(2)} - \lambda}{\lambda_h^{(3)} - \lambda} \right| = 589 \approx 2^{9.2}.$$

(7.9)

From Table 4, we see the following approximate convergence rate for $\lambda_{h,p}$,

$$\left| \frac{\lambda_{h,p}^{(1)} - \lambda}{\lambda_{h,p}^{(2)} - \lambda} \right| = 213 \approx 2^{7.7}, \quad \left| \frac{\lambda_{h,p}^{(2)} - \lambda}{\lambda_{h,p}^{(3)} - \lambda} \right| = 589 \approx 2^{9.2}.$$

(7.10)

Fig. 9. The contours of $u_h(x)$ at the target points $\left(\|u_h\|_{L^2}, \lambda_h\right) = (1, \lambda_h)$ of the (a) first, (b) second, (c) third, and (d) fourth solution branches of Eq. (1.6) with $v_1 = v_2 = 100$, $\mu = 10$, and $d_1 = d_2 = 1/3$. 

The first extrapolation results for the minimal eigenvalue in (7.8) and (7.9) are two order higher than the results in (7.2). Note that the extrapolation approximation \( \frac{\lambda^2_{2p-1}}{\lambda^2_{2p}} \approx 19.73920880218252 \) at \( N = 24 \) in Table 4 is correct up to 12 significant digits for the minimal eigenvalue \( 2\pi^2 = 19.739208802178717 \), and the second extrapolation result in (7.9) for the minimal eigenvalue \( 2\pi \) has order of convergence \( O(h^4) \).

**Example 2 (Model I (Dirichlet and periodic boundary conditions)).**

\[-\Delta u = \lambda u \quad \text{in} \ S, \]
\[u(x, 1) = u(x, 0) = 0, \quad u(0, y) = u(1, y), \quad u_0(0, y) = u_0(1, y).\]

The numerical results are listed in Table 5. To see the order of convergence clearly, we also list the ratio of errors in Table 6 based on Table 5. From Tables 5 and 6, we see the following asymptotic rates for the minimal eigenvalue \( \lambda_1 = \pi^2 = 9.869604401089358 \) and the eigenfunction corresponding to \( \lambda_1 \).

\[
\frac{\lambda^{(1)}_{2p} - \lambda_1}{\lambda^{(2)}_{2p} - \lambda_1} = 898 \approx 2^{9.8}, \quad \frac{\lambda^{(3)}_{2p} - \lambda_1}{\lambda^{(4)}_{2p} - \lambda_1} = 1350 \approx 2^{10.4}.
\] (7.9)

The results are listed in Tables 7 and 8. The numerical rates for \( \|u - u_h\|_{0.5} = O(h^5) \) and \( \|u - u_h\|_{1.5} = O(h^6) \) in (7.13) also are one order higher than those of the original order \( O(h^5) \) and \( O(h^6) \) in [34]. Moreover, we used the a posteriori interpolant to obtain the superconvergence \( O(h^5) \) and \( O(h^6) \) in the \( L_2 \) and \( H_1 \) norms in (7.14), respectively.

Using the results in Table 5, we may obtain more accurate approximations by the extrapolation formula (7.6) and (7.7). The details are omitted.

**Example 3 (Model II (Neumann and periodic boundary conditions)).**

\[-\Delta u = \lambda u \quad \text{in} \ S, \]
\[u(x, 1) = u(x, 0) = 0, \quad u(0, y) = u(1, y), \quad u_0(0, y) = u_0(1, y).\]

The results are listed in Tables 7 and 8. The numerical rates for Model II are also exactly the same as those for the Dirichlet bound-

\[
|\lambda_h - \lambda_1| = O(h^5), \quad |\lambda_h - \lambda_1| = O(h^8),\]

\[
|u - u_h|_{0.5} = O(h^4), \quad \|u - u_h\|_{1.5} = O(h^5),\]

\[
|u - u_h|_{0.5} = O(h^5), \quad \|u - u_h\|_{1.5} = O(h^6),\]

\[
|u - u_h|_{0.5} = O(h^5), \quad \|u - u_h\|_{1.5} = O(h^6).\]

(7.11) (7.12) (7.13) (7.14)

Note that the extrapolation approximation \( \lambda^{(2)}_{2p} = 19.73920880218252 \) in (5.17) by the theoretical estimation. The numerical rates of superclose \( \|u - u_h\|_{0.5} = O(h^5) \) and \( \|u - u_h\|_{1.5} = O(h^6) \) in (7.13) also are one order higher than those of the optimal order \( O(h^5) \) and \( O(h^6) \) in [34]. Moreover, we used the a posteriori interpolant to obtain the superconvergence \( O(h^5) \) and \( O(h^6) \) in the \( L_2 \) and \( H_1 \) norms in (7.14), respectively.
ary condition and Model I. Moreover, they are completely in accordance with the theoretical results in Theorems 5.1 and 5.3, by noting that \(|\lambda_4 - \lambda_1| = O(h^3)|.

**Example 4** (The SEP with trapping potential). We impose the trapping potential \(V(x,y) = \frac{1}{1 + x^2 + y^2}\) on (7.10) and obtain

\[ -\Delta u - lu + V(x,y)u = 0 \quad \text{in} \quad \mathcal{S}. \]

where \(\mathcal{S} = \{(x,y): 0 \leq x \leq 1, 0 \leq y \leq 1\}. The first five eigenvalues of (7.16) are listed in Table 9.

**Example 5** (Energy levels and superfluid densities of a 2D BEC in a potential condition). We discretized (1.6) using the Adini elements with \(h = \frac{1}{64}\). Here we imposed the additional normal derivative boundary conditions

\[ \frac{\partial u}{\partial n} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} \bigg|_{x=0} = 0, \quad u(0,y) = u(1,y), \quad u_0(0,y) = u_0(1,y), \]

where \(S = \{(x,y): 0 \leq x \leq 1, 0 \leq y \leq 1\}. The first five eigenvalues of (7.16) are listed in Table 9.

8. Concluding remarks

Based on our discussions given above, let us give some concluding remarks concerning the performance of the Adini elements for the Schrödinger equations.

(1) We have studied the Adini elements for the Schrödinger equation with periodic boundary conditions. The convergence rate \(O(h^6)\) has been obtained for the linear eigenvalue problem using the standard FEM analysis in [22,21]. The mixed type boundary conditions involving periodic boundary condition, namely, Models I and II, have been considered. The superconvergence rate \(O(h^8)\) have been derived, where \(q = 1\) and \(q = 2\) for Models I and II, respectively.

(2) In [25,26,28,27], superconvergence of FEM are studied only for the Dirichlet condition. The periodical boundary conditions are explored for the superconvergence of FEM in this paper.

(3) The numerical results in Section 7 shows that the superconvergence for the leading eigenvalue is \(O(h^5)\). Moreover, higher convergence rates can be achieved by the extrapolation formulas in (7.6) and (7.7). To the best of our knowledge, such results are the most accurate numerical solutions by FEM ever published.

(4) The Adini elements combined with the two-grid scheme [17] can be applied to compute the first few leading eigenpairs of the Schrödinger–Poison eigenvalue problem and the Schrödinger–Poison system. The details will be given elsewhere.

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References


